

Antiholomorphic Dynamics: Topology of Parameter Spaces and Discontinuity of Straightening

by

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Abstract

The goal of this thesis is to study the dynamics of unicritical antiholomorphic polynomials $\bar{z}^d + c$, and to explore the combinatorial and topological properties of the multicorns, which are the connectedness loci of the maps under consideration. We, on one hand, prove many topological differences between the multicorns and their holomorphic counterparts, the multibrot sets, and on the other hand, study the self-similarity property of the multicorns.

We develop a combinatorial theory of orbit portraits of unicritical antiholomorphic polynomials, following Milnor's work on orbit portraits for quadratic polynomials. There is an explicit description of the orbit portraits that can occur for our maps in terms of their characteristic angles, which turns out to be rather restricted when compared with the holomorphic case. We also define 'formal orbit portraits', and prove a realization theorem for these combinatorial objects.

In a joint work with Dierk Schleicher and Shizuo Nakane, we study the boundaries of hyperbolic components, and the bifurcation structure in the multicorns. The even period hyperbolic components turn out to be uninteresting since they are similar to the holomorphic case, but the odd period hyperbolic components are strikingly different: their boundaries only consist of parabolic parameters, many of which are quasi-conformally conjugate to each other. We also make use of the theory of orbit portraits to prove a discontinuity phenomenon for the landing points of dynamical rays, and to count the number of hyperbolic components of a given period in the multicorns.

One of the main problems in studying the multicorns is that the parameter dependence of antiholomorphic polynomials is only real-analytic, thus making it difficult to use complex-analytic tools directly. To circumvent this problem, we construct a one complex-dimensional quasi-conformal deformation of the persistently parabolic maps, and then use complex-analytic tools. In particular, this construction yields a holomorphic motion of the Julia sets of the persistently parabolic antiholomorphic polynomials, which allows us to apply results from thermodynamic formalism to show that the Hausdorff dimensions of the Julia sets vary real-analytically on the boundaries of the odd period hyperbolic components.

We show that the parameter rays accumulating on the boundaries of odd

period (except period one) hyperbolic components of the multicorns do not land; they non-trivially accumulate on intervals of parabolic parameters. This is in stark contrast with the corresponding situation for the multibrot sets, where all rational parameter rays are known to land. The proof involves the theory of perturbation of parabolic points as developed by Hubbard and Schleicher, and a study of some geometric properties of the Fatou coordinates. This is a joint work with Hiroyuki Inou.

Using methods similar to the ones mentioned in the previous paragraph, we show that the centers of the hyperbolic components of the multicorns are not dense on the boundary of the multicorns, and the harmonic measure of the exterior of the multicorn does not equi-distribute the centers. This, once again, is a major topological difference between the multicorns and the multibrot sets.

Finally, we study the behavior of the ‘straightening map’ from the ‘baby multicorns’ to the original multicorns. More precisely, we prove discontinuity of the straightening map (for even degree multicorns) at explicit parameters. This is the first known example where the straightening map fails to be continuous on a real two-dimensional slice of a holomorphic family of holomorphic polynomials. The proof of discontinuity of the straightening map is carried out by showing that all non-real ‘umbilical cords’ of the multicorns ‘wobble’. This generalizes a theorem of Hubbard and Schleicher, and settles a conjecture of Hubbard, Inou, Milnor and Schleicher.

There are four appendices. The first one discusses the landing behavior of umbilical cords for the tricorn-like sets in the parameter space of real cubic polynomials. More precisely, we prove that all non-real umbilical cords for these tricorn-like sets wobble, and that the corresponding straightening maps are discontinuous. The second appendix is a collection of some conformal conjugacy invariants that one can associate with attracting or parabolic basins containing several critical orbits. The third appendix addresses some local-global questions for polynomial parabolic germs; in particular, we show that two conformally distinct unicritical holomorphic polynomials each having a parabolic cycle with a single petal (at each parabolic point) cannot have conformally conjugate parabolic germs. We also show that one can recover the non-cusp odd period parabolic parameters of the multicorns, up to some natural rotational and reflection symmetries, from their parabolic germs. The final appendix deals with some examples of the dynamically defined algebraic sets $\text{Per}_n(1)$ (in various families of polynomials), the nature of their singularities, and the ‘dynamical’ behavior of these singular parameters.

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Statutory Declaration
(on Authorship of a Dissertation)

I, Sabyasachi Mukherjee, hereby declare that I have written this PhD thesis independently, unless where clearly stated otherwise. I have used only the sources, the data and the support that I have clearly mentioned. This PhD thesis has not been submitted for conferral of degree elsewhere.

I confirm that no rights of third parties will be infringed by the publication of this thesis.

Bremen, June 24, 2015

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Chapter 1

Introduction

1.1 Background on Antiholomorphic Dynamics

Holomorphic dynamics has been one of the most exciting and active areas of mathematical research in the past few decades. Since the seminal works of Douady and Hubbard on the dynamics of quadratic polynomials, a great deal of research has been done to understand the structure of the Mandelbrot set. Since quadratic polynomials only have a single critical point, the corresponding parameter space is the complex plane, and the holomorphic parameter dependence of the polynomials makes it possible to use the powerful machinery of one variable complex analysis. The situation becomes far more complex when one looks at the parameter spaces of higher degree polynomials. The parameter spaces are now higher dimensional, and pictorial intuitions are harder to obtain. Moreover, the presence of several critical orbits and their mutual interactions allow for much more complicated dynamical configurations. Due to these obstacles, the understanding of parameter spaces of higher degree polynomials is far from complete.

In a pioneering paper on the parameter space of cubic polynomials, Milnor classified the possible types of hyperbolic components, and made several computer pictures [Mil92]. In this paper, he depicts a curious three-cornered object (which he named the ‘tricorn’, see Figure 1.1) in a real two-dimensional slice of the parameter space where the critical orbits of the polynomials possess a certain dynamical and involutive symmetry. The dynamics of the corresponding maps exhibit a quadratic anti-polynomial-like behavior, and this was the primary motivation to view the parameter spaces of quadratic antiholomorphic polynomials as a prototypical object in the study of iterated real cubic polynomials, and more generally, of real forms of holomorphic maps.

In this thesis, we carry out a systematic study of some combinatorial and topological properties of the iteration of unicritical antiholomorphic polynomials $f_c(z) = \bar{z}^d + c$, for any degree $d \geq 2$, and $c \in \mathbb{C}$, and of their parameter spaces. In the sequel, antiholomorphic polynomials will be called

‘anti-polynomials’ for the sake of brevity. Any unicritical anti-polynomial can be affinely conjugate to an anti-polynomial of the form f_c .

Let us begin with the definitions of some dynamically defined invariant sets that we would be interested in.

Definition 1.1.1. • *The filled-in Julia set $K(f_c)$ of f_c is defined as the set of all points in $\hat{\mathbb{C}}$ which remain bounded under all iterations of f_c .*

- *The basin of infinity $\mathcal{A}_\infty(f_c)$ of f_c is defined as the set of all points in $\hat{\mathbb{C}}$ whose forward iterates under f_c converge to ∞ .*
- *The boundary of the filled-in Julia set $K(f_c)$ is called the Julia set $J(f_c)$ of f_c .*
- *The complement of the Julia set $J(f_c)$ (in $\hat{\mathbb{C}}$) is defined to be the Fatou set $F(f_c)$ of f_c .*

This leads, as in the holomorphic case, to the notion of the *Connectedness Locus* of degree d unicritical anti-polynomials:

Definition 1.1.2 (Multicorns). *The multicorn of degree d is defined as $\mathcal{M}_d^* := \{c \in \mathbb{C} : K(f_c) \text{ is connected}\}$. The multicorn of degree 2 is called the tricorn.*

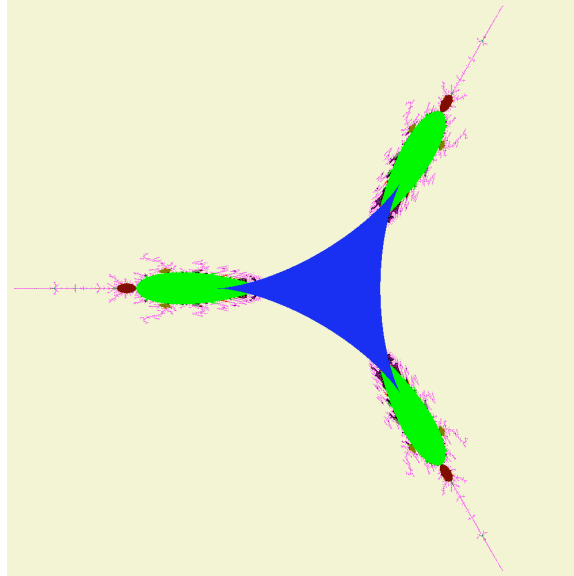


Figure 1.1: The tricorn, where the central deltoid region is its period 1 hyperbolic component.

The importance of the study of unicritical anti-polynomials and their parameter spaces stems from many perspectives.

- They are interesting dynamical systems in their own right. Although they bear similarities with the dynamics of ordinary polynomials, there are some surprising dissimilarities as well: for example the combinatorics of the rays landing at a periodic point of odd period is quite restricted, and any indifferent periodic point of odd period is necessarily parabolic. At a combinatorial level, this demands a better understanding of the map $\vartheta \mapsto -d\vartheta$, which reflects the orientation reversing property of the dynamics.
- Antiholomorphic dynamics appears naturally as prototypical dynamics in the study of iterated real cubic polynomials, a fact first observed by John Milnor [Mil92]. Milnor found small tricorn-like sets in the parameter space of real cubic polynomials with ‘bitransitive’ mapping scheme such that the critical orbits of the polynomials are symmetric with respect to the real line. He also observed multicorn-like sets in other families of rational maps with two critical points [Mil00a]. The dynamics of the corresponding maps exhibit a quadratic anti-polynomial-like behavior, and this was the primary motivation to view the parameter spaces of quadratic anti-polynomials as a prototypical object in the study of iteration of rational maps with two critical points. More generally, one expects the existence of multicorn-like sets in any family of polynomials or rational maps with (at least) two critical orbits such that a pair of critical orbits are symmetric with respect to an antiholomorphic involution. Numerical evidences of this fact can be found in the recent works on the parameter spaces of certain families of rational maps, such as the family of antipode preserving cubic rationals [BBM15], Blaschke products [CFG15], etc.
- The family of unicritical anti-polynomials of a fixed degree can be embedded as a real two-dimensional slice in the space of some family of holomorphic polynomials of higher degree. For example, the second iterate of the anti-polynomial $f_c(z) = \bar{z}^2 + c$ is the holomorphic map $f_c^\circ = (z^2 + \bar{c})^2 + c$. Thus the family of quadratic anti-polynomials sit as a real two-dimensional $(a = \bar{b})$ slice in the space of biquadratic complex polynomials $(z^2 + a)^2 + b$. Thus, many topological properties of higher-dimensional parameter spaces (e.g. non local-connectedness [HS14]) are reflected in the parameter spaces of unicritical anti-polynomials. So a complete knowledge of the structure of the multicorns would contribute to a better understanding of the higher degree polynomial parameter spaces.
- The parameter dependence of the anti-polynomials is only real-analytic, so it is worthwhile to check how far certain analytical properties of the connectedness loci (eg. existence of baby Mandelbrot sets and baby

tricorn etc.) hold in absence of holomorphy.

The dynamics of quadratic antiholomorphic polynomials and its connectedness locus $\mathcal{M}_2^*$ were first studied in [CHRC89], and their numerical experiments showed major structural differences between the Mandelbrot set and the tricorn; in particular, they observed that there are bifurcations from the period 1 hyperbolic component to period 2 hyperbolic components along arcs in the tricorn, in contrast to the fact that bifurcations are always attached at a single point in the Mandelbrot set. The bifurcation structure in the family of quadratic antiholomorphic polynomials was studied in [Win90]. However, as remarked earlier, it was Milnor who first observed the importance of the multicorns; he found little tricorn and multicorn-like sets as prototypical objects in the parameter space of real cubic polynomials [Mil92], and in the real slices of rational maps with two critical points [Mil00a]. Nakane [Nak93a] proved that the tricorn is connected, in analogy to Douady and Hubbard's classical proof on the Mandelbrot set. This generalizes naturally to multicorns of any degree. Later, Nakane and Schleicher, in [NS03], studied the structure of hyperbolic components of the multicorns via the multiplier map (even period case) and the critical value map (odd period case). These maps are branched coverings over the unit disk of degree $d - 1$ and $d + 1$ respectively, branched only over the origin. Hubbard and Schleicher [HS14] proved that the multicorns are not pathwise connected, confirming a conjecture of Milnor. We will elucidate these results later in this section after having developed the requisite background.

To illustrate the possible problems with antiholomorphic parameter spaces, consider the family of antiholomorphic quadratic polynomials $P_\mu(z) = \mu\bar{z} + \bar{z}^2$ which was discussed in the introduction of [NS03]; each P_μ is conformally conjugate to some f_c . As shown in Figure 1.2, the circle $|\mu| = 1$ consists of maps with parabolic fixed points, and most parameters on this circle have neighborhoods in which every parameter has an attracting or indifferent fixed point: the open mapping principle for the multiplier map fails in this parametrization! This is related to a different problem of the parametrization: within the family $\{f_c\}_{c \in \mathbb{C}}$, each map f_c is conformally conjugate to two other maps (with parameters ζc and $\zeta^2 c$, where ζ is any third root of unity). However, in the family $\{P_\mu\}_{\mu \in \mathbb{C}}$, each map is conjugate to three or two further maps in the same family, depending on whether or not there is an attracting fixed point (in both cases, the exceptional case $c = 0$ respectively $\mu = 0$ behaves differently).

Antiholomorphic polynomials of the form $q_\lambda(z) = \lambda(1 + \bar{z}/d)^d$ form the “true” parameter space of our maps: every f_c is conformally conjugate to one and only one antiholomorphic polynomial q_λ ; it satisfies $\lambda = d\bar{c}^d/c$. The map $c \mapsto \lambda = d\bar{c}^d/c$ is a real-analytic branched cover of degree $d + 1$, ramified only over $c = \lambda = 0$. We will use this parametrization in some of our proofs. The multicorn $\mathcal{M}_d^*$ has a $d + 1$ -fold rotational symmetry, and form a $d + 1$ -fold branched cover over the true parameter space. The quotient of $\mathcal{M}_d^*$ by this symmetry is thus naturally called the “unicorn”. Pictorial illustrations of

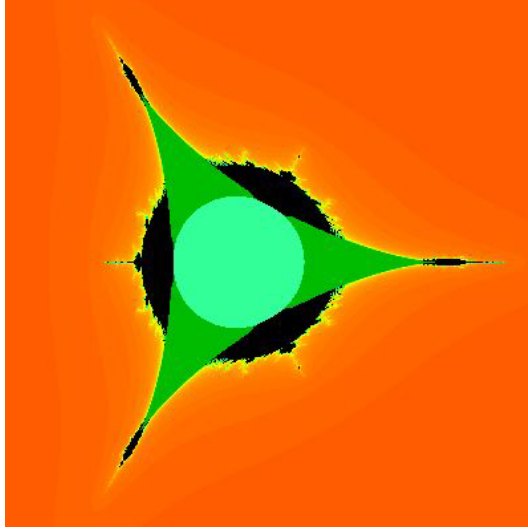


Figure 1.2: The connectedness locus of polynomials $\mu\bar{z} + z^2$. The circle in the center consists of parameters with parabolic fixed points, and it intersects the boundaries of four distinct hyperbolic components of period 1: the disk inside, and three symmetric components outside of the circle. In this space, Theorem 3.1.1 is false.

these fractals and their symmetries can be found in [LS96] and [NS03].

We will now briefly discuss the concept of multipliers for odd and even periodic cycles of an antiholomorphic map.

Let f be an antiholomorphic map. Let $\mathcal{O}_{\text{odd}} = \{z_0, z_1, \dots, z_{k-1}\}$ be a periodic cycle of *odd* period k of f . We define the antiholomorphic multiplier of f at z_0 (the $\bar{z}$ -derivative of the antiholomorphic first return map) as $\lambda_0 := \frac{\partial f_c^k}{\partial \bar{z}}(z_0)$ (with respect to a fixed coordinate system). One easily computes that,

$$\frac{\partial f_c^k}{\partial \bar{z}}(z_0) = \frac{\partial f}{\partial \bar{z}}(z_0) \overline{\frac{\partial f}{\partial \bar{z}}(z_1)} \frac{\partial f}{\partial \bar{z}}(z_2) \overline{\frac{\partial f}{\partial \bar{z}}(z_3)} \cdots \frac{\partial f}{\partial \bar{z}}(z_{k-1}).$$

This shows that the antiholomorphic multiplier is different at different points of the cycle. Also, the antiholomorphic multiplier is not a conformal invariant; an easy computation in power series shows that it is preserved by change of coordinates which are tangent to the identity at the periodic point, but is rotated otherwise. One can also define the corresponding holomorphic multiplier as the z -derivative of the holomorphic second return map. The following computation shows the relation between the holomorphic and the antiholomorphic multipliers at z_0 .

$$\begin{aligned}
\frac{\partial f^{\circ 2k}}{\partial z}(z_0) &= \frac{\partial f^{\circ k}}{\partial \bar{z}}(z_0) \overline{\frac{\partial f^{\circ k}}{\partial \bar{z}}(z_0)} \\
&= \left| \frac{\partial f^{\circ k}}{\partial \bar{z}}(z_0) \right|^2 \\
&= |\lambda|^2.
\end{aligned}$$

The holomorphic multiplier is the same at each point of the cycle $\mathcal{O}_{\text{odd}}$, is a conformal invariant. It follows that the absolute value of the antiholomorphic multiplier is a conformal invariant of the cycle.

Let us now turn our attention to cycles of *even* period. As before, let f be an antiholomorphic map, $k \in \mathbb{N}$, and $\mathcal{O}_{\text{even}} = \{z_0, z_1, \dots, z_{2k-1}\}$ be a periodic cycle of f . Since the first return map of each point of the cycle is now holomorphic, one can only make sense of the holomorphic multiplier. However, the cycle $\mathcal{O}_{\text{even}}$ breaks into two separate k -cycles under the holomorphic second iterate $f^{\circ 2}$. Since the holomorphic multiplier is same at each point of a cycle, it suffices to compute it at z_0 and $f^{\circ k}(z_0)$.

$$\begin{aligned}
\frac{\partial f^{\circ 2k}}{\partial z}(z_0) &= \frac{\partial f^{\circ k}}{\partial \bar{z}}(f^{\circ k}(z_0)) \overline{\frac{\partial f^{\circ k}}{\partial \bar{z}}(z_0)} \\
&= \overline{\frac{\partial f^{\circ k}}{\partial \bar{z}}(z_0) \frac{\partial f^{\circ k}}{\partial \bar{z}}(f^{\circ k}(z_0))} \\
&= \overline{\frac{\partial f^{\circ 2k}}{\partial z}(f^{\circ k}(z_0))}.
\end{aligned}$$

Thus, the even periodic cycle $\mathcal{O}_{\text{even}}$ of f breaks into two separate cycles under $f^{\circ 2}$ and the holomorphic multipliers of these two cycles are complex conjugates of each other.

The local fixed point theory of holomorphic maps is classical [Mil06]. This carries over verbatim to the periodic cycles of even period of antiholomorphic maps, since the first return map of such maps is holomorphic. We now recall the existence of local uniformizing coordinates for (super-)attracting and parabolic cycles of odd period of antiholomorphic maps. We begin with the case of attracting cycles.

For attracting (but not super-attracting) cycles of odd period, there is a conformal change of coordinates (known as Koenigs coordinate) φ for f at z_0 such that locally near z_0 , $\varphi \circ f^{\circ k}(z) = \overline{\lambda \varphi(z)}$. This was proved in [NS03, Lemma 5.1].

Lemma 1.1.3 (Antiholomorphic Koenigs coordinates). *There is a neighborhood U of z_0 and a conformal isomorphism $\varphi : U \rightarrow \mathcal{D}$ such that $\varphi \circ f = b \cdot \bar{\varphi}$ on U , where the right-hand side is complex conjugation and multiplication with a complex number $b \in \mathbb{D}^* := \mathbb{D} \setminus \{0\}$.*

In other words, an antiholomorphic map is locally (anti-)linearizable at every attracting but not super-attracting fixed point. If g is the restriction of an anti-polynomial $f_c^{\circ n}$ with n odd, then we can and will choose U large enough so that a unique critical point w of $f_c^{\circ n}$ is on ∂U . Then φ extends to a homeomorphism $\varphi : \overline{U} \rightarrow \overline{\mathbb{D}}$, and it becomes unique if we require that $\varphi(w) = 1$. We define the critical value map of the attracting fixed point to be $c \mapsto \varphi(f_c^{\circ n}(w))$. This map takes images in $\mathbb{D}^*$, and is independent of coordinate changes. In the super-attracting case, we define the critical value map to be 0.

Using the multiplier map (for even period hyperbolic components) and the critical value map (for odd period hyperbolic components), Nakane and Schleicher managed to give real-analytic uniformizations (more precisely, real-analytic branched covers over the unit disk of degree $d-1$ and $d+1$ respectively, branched only over the origin) of the hyperbolic components of $\mathcal{M}_d^*$ proving that they are dynamically homeomorphic to the unit disc $\mathbb{D}$. They also proved the uniqueness of centers of the hyperbolic components.

Theorem 1.1.4 (Hyperbolic Components). *[NS03, §5] Every hyperbolic component of $\mathcal{M}_d^*$ is homeomorphic to $\mathbb{D}$, and has a unique center.*

The basin of infinity, and the corresponding Böttcher coordinate play a vital role in the dynamics of polynomials (compare [Mil06, §9, §18]). In the antiholomorphic setting, there is a parallel notion of uniformizing coordinates near ∞ which is due to Nakane.

Theorem 1.1.5 (Böttcher coordinates). *[Nak93a, Lemma 1] There is a conformal map φ_c near ∞ such that $\lim_{z \rightarrow \infty} \varphi_c(z)/z = 1$ and $\varphi_c \circ f_c(z) = \overline{\varphi_c(z)}^d$. φ_c extends as a conformal isomorphism to an equipotential containing 0, when $c \notin \mathcal{M}_d^*$, and extends as a biholomorphism from $\hat{\mathbb{C}} \setminus K(f_c)$ onto $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ when $c \in \mathcal{M}_d^*$.*

Using Theorem 1.1.5, one can define dynamical rays for anti-polynomials.

Definition 1.1.6 (Dynamical Ray). *The dynamical ray $R_c(t)$ of f_c at an angle t is defined as the pre-image of the radial line at angle t under φ_c .*

The dynamical ray $R_c(t)$ at angle $t \in \mathbb{R}/\mathbb{Z}$ maps to the dynamical ray $R_c(-dt)$ at angle $-dt$ under f_c . We say that the dynamical ray $R_c(t)$ of f_c lands if $\overline{R_c(t)} \cap K(f_c)$ is a singleton, and this unique point, if exists, is called the landing point of $R_c(t)$. It is worth mentioning that for a complex polynomial (of degree d) with connected Julia set, every dynamical ray at a periodic angle (under multiplication by d) lands at a repelling or parabolic periodic point, and conversely, every repelling or parabolic periodic point is the landing point of at least one periodic dynamical ray [Mil06, §18]. Since the second iterate of

an anti-polynomial is a complex polynomial, these facts remain true for anti-polynomials as well. We refer the readers to [NS03, §3] for further properties of dynamical rays of anti-polynomials.

The next result, which is due to Nakane [Nak93a], utilizes the existence of Böttcher coordinates near ∞ for anti-polynomials, and demonstrates the connectedness of the multicorns.

Theorem 1.1.7 (Connectedness of The Multicorns). *The map $\Phi : \mathbb{C} \setminus \mathcal{M}_d^* \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$, defined by $c \mapsto \varphi_c(c)$ (where φ_c is the Böttcher coordinate of f_c near ∞) is a real-analytic diffeomorphism. In particular, the multicorns are connected.*

In principle, the proof of this theorem is analogous to Douady and Hubbard's proof of connectedness of the Mandelbrot set. However, due to the lack of holomorphic parameter dependence of the Böttcher coordinates, one needs to resort to different techniques to show that the candidate map is indeed a diffeomorphism.

The previous theorem also allows us to define parameter rays of the multicorns.

Definition 1.1.8 (Parameter Ray). *The parameter ray at angle ϑ of the multicorn $\mathcal{M}_d^*$, denoted by $\mathcal{R}_\vartheta^d$, is defined as $\{\Phi^{-1}(re^{2\pi i\vartheta}) : r > 1\}$, where Φ is the real-analytic diffeomorphism from the exterior of $\mathcal{M}_d^*$ to the exterior of the closed unit disc in the complex plane constructed in Theorem 1.1.7.*

Definition 1.1.9 (External Angle of a Parameter). *Let $c \in \mathbb{C} \setminus \mathcal{M}_d^*$. Then the unique angle $t(c)$ such that $c \in \mathcal{R}_{t(c)}^d$ is called the external angle of the parameter c .*

Remark 1.1.10. *Some comments should be made on the definition of the parameter rays. Observe that unlike the multibrot sets (these are the connectedness loci of unicritical holomorphic polynomials $z^d + c$ [EMS14]), the parameter rays of the multicorns are not defined in terms of the Riemann map of the exterior. In fact, the Riemann map of the exterior of $\mathcal{M}_d^*$ has no obvious dynamical meaning. We have defined the parameter rays via a dynamically defined diffeomorphism of the exterior of $\mathcal{M}_d^*$, and it is easy to check that this definition of parameter rays agree with the notion of stretching rays (see [KN04] for the definition of stretching rays) in the family of polynomials $(z^d + a)^d + b$.*

Before moving on to the local theory of indifferent dynamics, let us recall a few basic definitions which will be useful in this thesis.

Definition 1.1.11 (Rational Lamination). *The rational lamination of an anti-polynomial f_c is defined as an equivalence relation on $\mathbb{Q}/\mathbb{Z}$ such that $\vartheta_1 \sim \vartheta_2$ if and only if the dynamical rays $R_c(\vartheta_1)$ and $R_c(\vartheta_2)$ land at the same point of $J(f_c)$. It is denoted by $\mathcal{RL}(f_c)$.*

Definition 1.1.12 (Accumulation Set of a Ray). *The accumulation set of a parameter ray $\mathcal{R}_\vartheta^d$ of $\mathcal{M}_d^*$ is defined as $L_{\mathcal{M}_d^*}(\vartheta) := \overline{\mathcal{R}_\vartheta^d} \cap \mathcal{M}_d^*$.*

Definition 1.1.13 (Impression of a Ray). *The impression of a parameter ray $\mathcal{R}_\vartheta^d$ of $\mathcal{M}_d^*$ is defined as the set of all possible limits $I_{\mathcal{M}_d^*}(\vartheta) := \lim_{r \rightarrow 1, \varphi \rightarrow \vartheta} \Phi^{-1}(re^{2\pi i \varphi})$, where Φ is the real-analytic diffeomorphism from the exterior of $\mathcal{M}_d^*$ to the exterior of the closed unit disc in the complex plane constructed in Theorem 1.1.7.*

We have observed that the holomorphic multiplier of an odd periodic cycle of an antiholomorphic map is real and positive. Therefore, every indifferent cycle of odd period of an antiholomorphic map must be parabolic with holomorphic multiplier $+1$. Hence, it suffices to discuss the local picture for parabolic fixed points. In holomorphic dynamics, the local dynamics in attracting petals of parabolic periodic points is well-understood: there is a local coordinate ζ which conjugates the first-return dynamics to the form $\zeta \mapsto \zeta + 1$ in a right half plane (see Milnor [Mil06, Section 10] or Carleson-Gamelin [CG93, Section II.5]). Such a coordinate ζ is called a *Fatou coordinate*. Thus the quotient of the petal by the dynamics is isomorphic to a bi-infinite cylinder, called an *Ecalte cylinder*. Note that Fatou coordinates are uniquely determined up to addition of a complex constant.

In antiholomorphic dynamics, an additional structure is given by the antiholomorphic intermediate iterate, and there is a natural choice for a preferred Fatou coordinate. This was proved in [HS14, Lemma 2.3].

Lemma 1.1.14 (Antiholomorphic Fatou Coordinates). *Suppose z_0 is a parabolic periodic point of odd period k of f_c with only one petal and U is a periodic Fatou component with $z_0 \in \partial U$. Then there is an open subset $V \subset U$ with $z_0 \in \partial V$ and $f_c^{\circ k}(V) \subset V$ so that for every $z \in U$, there is an $n \in \mathbb{N}$ with $f_c^{\circ nk}(z) \in V$. Moreover, there is a univalent map $\Phi: V \rightarrow \mathbb{C}$ with $\Phi(f_c^{\circ k}(z)) = \overline{\Phi(z)} + 1/2$, and $\Phi(V)$ contains a right half plane. This map Φ is unique up to horizontal translation.*

The map Φ will be called an *antiholomorphic Fatou coordinate* for the petal V ; it satisfies $\Phi(f_c^{\circ 2k}(z)) = \Phi(z) + 1$ for $z \in V$ in accordance with the standard theory of holomorphic Fatou coordinates. This lemma applies more generally to antiholomorphic indifferent periodic points such that the attracting petal has odd period.

If c is a cusp, the period of U is even. Indeed, if c is a cusp, then the parabolic periodic point z_0 has two petals, and hence is a dynamical root of U . By [NS03, Corollary 4.2], if the period of U under f_c were odd, then z_0 would lie on the boundary of only one Fatou component, which contradicts the fact that z_0 has two petals. It also follows from [NS03, Corollary 4.2] that the period of U under f_c is $2k$. Therefore, the first return map of U is $f_c^{\circ 2k}$, which is holomorphic, and the previous lemma does not apply.

The antiholomorphic iterate interchanges both ends of the Ecalle cylinder, so it must fix one horizontal line around this cylinder (the *equator*). The change of coordinate has been so chosen that the equator is the projection of the real axis. We will call the vertical Fatou coordinate the *Ecalle height*. Its origin is the equator.

The following lemma shows the abundance of parabolic parameters on the boundaries of odd period hyperbolic components of the multicorns. See [HS14, Lemma 2.2] for a proof.

Lemma 1.1.15 (Indifferent Dynamics of Odd Period). *The boundary of a hyperbolic component of odd period k consists entirely of parameters having a parabolic orbit of period k . In local conformal coordinates, the $2k$ -th iterate of such a map has the form $z \mapsto z + z^{q+1} + \dots$ with $q \in \{1, 2\}$.*

Definition 1.1.16 (Parabolic Cusps). *A parameter c will be called a cusp point if it has a parabolic periodic point of odd period such that $q = 2$ (equivalently, if every parabolic periodic point of c has exactly two attracting petals) in the previous lemma. Odd periodic parabolic parameters with $q = 1$ (equivalently, with exactly one petal at each parabolic periodic point) will be called non-cusp parabolic parameters.*

Definition 1.1.17 (Characteristic Components and Points). *The characteristic Fatou component of f_c is defined as the unique Fatou component of f_c containing the critical value c . If f_c has a parabolic periodic orbit, then its characteristic parabolic point is defined as the unique parabolic point lying on the boundary of the characteristic Fatou component.*

Definition 1.1.18 (Roots and Co-Roots of Fatou Components). *Let z be a boundary point of a periodic Fatou component U corresponding to a (super-)attracting or parabolic unicritical antiholomorphic polynomial f_c so that the first return map of U fixes z . Then we call z a root of U if it disconnects the filled-in Julia set; if it does not, we call it a co-root.*

It was observed by Nakane [Nak93a] that every non-cusp parabolic parameter on the boundary of a hyperbolic component H of odd period is quasiconformally conjugate to nearby parameters on ∂H [Nak93b, Theorem 2.15]. We will give a refined and sharper version of this result in Theorem 3.2.1.

For an isolated fixed point $\hat{z} = f(\hat{z})$ where $f : U \rightarrow \mathbb{C}$ is a holomorphic function on a connected open set $U \subset \mathbb{C}$, the residue fixed point index of f at $\hat{z}$ is defined to be the complex number

$$\iota(f, \hat{z}) = \frac{1}{2\pi i} \oint \frac{dz}{z - f(z)}$$

where we integrate in a small loop in the positive direction around $\hat{z}$. If the multiplier $\rho := f'(\hat{z})$ is not equal to $+1$, then a simple computation shows

that $\iota(f, \hat{z}) = 1/(1 - \rho)$. If z_0 is a parabolic fixed point with multiplier $+1$, then in local holomorphic coordinates the map can be written as $f(w) = w + w^{q+1} + \alpha w^{2q+1} + \dots$ (putting $\hat{z} = 0$), and α is a conformal invariant (in fact, it is the unique formal invariant other than q : there is a formal, not necessarily convergent, power series that formally conjugates f to the first three terms of its series expansion). A simple calculation shows that α equals the parabolic fixed point index. The ‘résidu itératif’ of f at the parabolic fixed point $\hat{z}$ of multiplier 1 is defined as $\left(\frac{q+1}{2} - \alpha\right)$. It is easy to see that the fixed point index does not depend on the choice of complex coordinates, and is a conformal invariant (compare [Mil06, §12]).

The typical structure of the boundaries of hyperbolic components of even periods is that there are $d - 1$ isolated root or co-root points which are connected by curves off the root locus. As we shall see later (in Chapter 3), for hyperbolic components of odd periods, the story is in a certain sense just the opposite: the analogues of roots or co-roots are now arcs, of which there are $d + 1$, and the analogues of the connecting curves are the isolated cusp points between the arcs; see Theorem 1.2.2. There is trouble, of course, where components of even and odd periods meet, and we get bifurcations along arcs: the root of the even period component stretches along parts of two arcs. This phenomenon was first observed in [CHRC89] for the main component of the tricorn. The precise statement is given in the following results, which were proved in [HS14, Proposition 3.7, Theorem 3.8, Corollary 3.9]. The proof of this fact uses the concept of holomorphic fixed point index. The main idea is that when several simple fixed points merge into one parabolic point, each of their indices tends to ∞ , but the sum of the indices tends to the index of the resulting parabolic fixed point, which is finite.

Lemma 1.1.19 (Fixed Point Index on Parabolic Arc). *Along any parabolic arc of odd period, the fixed point index is a real valued real-analytic function that tends to $+\infty$ at both ends.*

Theorem 1.1.20 (Bifurcations Along Arcs). *Every parabolic arc of period k intersects the boundary of a hyperbolic component of period $2k$ at the set of points where the fixed-point index is at least 1, except possibly at (necessarily isolated) points where the index has an isolated local maximum with value 1. In particular, every parabolic arc has, at both ends, an interval of positive length at which a bifurcation from a hyperbolic component of odd period k to a hyperbolic component of period $2k$ occurs.*

See Figure 1.4 (right), which is an enlargement of Figure 1.1.

We will need the concept of parabolic trees, which are defined in analogy with Hubbard trees for post-critically finite polynomials. Our definition will follow [HS14, Section 5].

Definition 1.1.21 (Parabolic Tree). *If c lies on a parabolic root arc of odd period k , we define a loose parabolic tree of f_c as a minimal tree within the*

filled-in Julia set that connects the parabolic orbit and the critical orbit, so that it intersects the critical value Fatou component along a simple $f_c^{\circ k}$ -invariant curve connecting the critical value to the characteristic parabolic point, and it intersects any other Fatou component along a simple curve that is an iterated preimage of the curve in the critical value Fatou component. Since the filled-in Julia set of a parabolic polynomial is locally connected and hence path connected, any loose parabolic tree connecting the parabolic orbit is uniquely defined up to homotopies within bounded Fatou components. It is easy to see that the parabolic tree intersects the Julia set in a Cantor set, and these points of intersection are the same for any loose tree (note that for simple parabolics, any two periodic Fatou components have disjoint closures).

By construction, the forward image of a loose parabolic tree is again a loose parabolic tree. A simple standard argument (analogous to the post-critically finite case) shows that the boundary of the critical value Fatou component intersects the tree at exactly one point (the characteristic parabolic point), and the boundary of any other bounded Fatou component meets the tree in at most d points, which are iterated pre-images of the characteristic parabolic point [Sch00, Lemma 3.5] [EMS14, Lemma 3.2, Lemma 3.3]. The critical value is an endpoint of the parabolic tree. All branch points of the parabolic tree are either in bounded Fatou components or repelling (pre-)periodic points; in particular, no parabolic point (of odd period) is a branch point.

Let us now come to the most important technical ingredient in the study of the parameter spaces of unicritical anti-polynomials. The theory of perturbation of parabolic points and parabolic implosion has been an extremely valuable tool in studying the parameter spaces of polynomials or rational maps in general, and a wealth of deep and non-trivial results have been proved with these machineries, e.g. discontinuous parameter dependence of Julia sets and their Hausdorff dimension, discontinuity of straightening, non-local connectedness of the cubic connectedness locus, etc. The existence of a preferred Fatou coordinate, or equivalently an intrinsic meaning to Ecalle height is a special feature of antiholomorphic maps, and contributes, surprisingly enough, to some simplification of matters, as compared to the holomorphic case (it automatically relates the heights of the attracting and repelling cylinders without a need for the *horn map* from the repelling back into the attracting cylinder). One place where this has been exploited is in the proof of non-local connectivity and non-path connectivity of the multicorns [HS14]. In fact, utilizing the notion of Ecalle heights, they developed a theory of perturbation of parabolic maps that is tailor-made for studying the parameter spaces of antiholomorphic maps. The tools developed in [HS14] will be repeatedly used in this thesis to good effect. The details of the perturbation theory is worked out in Section 5.1.

Theorem 1.1.22. [HS14] *For each $d \geq 2$, the multicorn $\mathcal{M}_d^*$ is not path connected.*

Polynomial-like maps developed by Douady and Hubbard [DH85b] make sense also in the antiholomorphic setting.

Definition 1.1.23 (Polynomial-like and Anti-polynomial-like Maps). *We call a map $g : U' \rightarrow U$ polynomial-like (respectively anti-polynomial-like) if*

- U', U are topological disks in $\mathbb{C}$, and $\overline{U'} \subset U$.
- $g : U' \rightarrow U$ is holomorphic (respectively antiholomorphic), and proper.

The filled-in Julia set $K(g)$ and the Julia set $J(g)$ are defined as follows:

$$K(g) = \{z \in U' : g^{\circ n}(z) \in U', \forall n \in \mathbb{N}\}, J(g) = \partial K(g).$$

In particular, we say that polynomial-like (or anti-polynomial-like) mapping is *unicritical-like* if it has a unique critical point of possibly higher multiplicity. The importance of (anti-)polynomial-like maps stems from the fact that they behave, in a certain sense, like (anti-)polynomials. This is justified by the following Straightening Theorem [DH85b, Theorem 1] which is proved in the same way as in the holomorphic case: every anti-polynomial-like map of degree d is hybrid equivalent to an anti-polynomial of equal degree.

Definition 1.1.24 (Hybrid Equivalence). *Two polynomial-like (or anti-polynomial-like) mappings $f : U' \rightarrow U$ and $g : V' \rightarrow V$ are hybrid equivalent if there exists a quasi-conformal homeomorphism $\varphi : U'' \rightarrow V''$ between neighborhoods U'' and V'' of $K(f)$ and $K(g)$ respectively, such that $\varphi \circ f = g \circ \varphi$ whenever both sides are defined, and $\bar{\partial}\varphi = 0$ almost everywhere in $K(f)$.*

Theorem 1.1.25 (Straightening Theorem). *Any polynomial-like (respectively anti-polynomial-like) mapping $g : U' \rightarrow U$ is hybrid equivalent to a holomorphic (respectively antiholomorphic) polynomial P of the same degree. Moreover, if $K(g)$ is connected, then P is unique up to affine conjugacy.*

Remark 1.1.26. *We define the degree of g as the number of pre-images of any point, so it is always positive. Hence, for an antiholomorphic map g , d is the degree (in the classical sense) of the proper holomorphic map $g^* : U \rightarrow V^*$ which is the complex conjugate of g .*

Let us illustrate how quadratic anti-polynomial-like behavior can be observed in the dynamical plane of a real cubic polynomial. Let $p(z) = z^3 - 3az + b$, $a, b \in \mathbb{R}$, $a < 0$ be a post-critically finite real cubic polynomial with a bitransitive mapping scheme (see [MP12] for the definition of mapping schemes). The condition $a < 0$ implies that the two critical points c_1 and c_2 of p are complex conjugate, and have complex conjugate forward orbits. The assumption on the mapping scheme implies that there exists an $n \in \mathbb{N}$ such that $p^{\circ n}(c_1) = c_2$, $p^{\circ n}(c_2) = c_1$ (compare Figure 1.3). If U is a sufficiently small neighborhood of the closure of the Fatou component containing c_1 , then

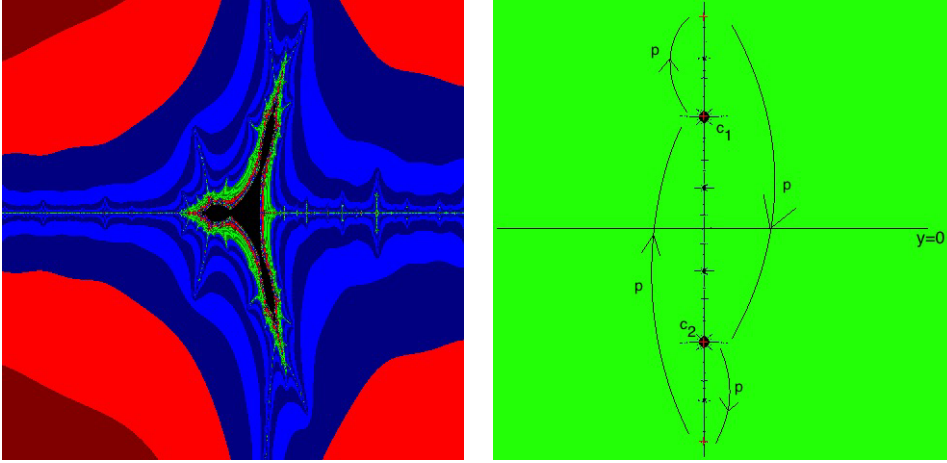


Figure 1.3: Left: A tricorn-like set in the parameter plane of real cubic polynomials. Right: The dynamical plane of a real cubic polynomial with real-symmetric critical orbits and a bitransitive mapping scheme.

we have, $\iota(\overline{U}) \subset p^{\circ n}(U)$ (where $\overline{U}$ is the topological closure of U , and ι is the complex conjugation map), i.e. $\overline{U} \subset (\iota \circ p^{\circ n})(U)$. Since $\iota \circ p^{\circ n}$ is an antiholomorphic map, we have an anti-polynomial-like map of degree 2 (with a connected filled-in Julia set) defined on U . The straightening Theorem 1.1.25 now yields a quadratic antiholomorphic map (with a connected filled-in Julia set) that is hybrid equivalent to $(\iota \circ p^{\circ n})|_U$. One can continue to perform this renormalization procedure as the real cubic polynomial p moves in the parameter space, and this defines a map from a suitable region in the parameter plane of real cubic polynomials to the tricorn. Similar considerations, applied to the centers of the odd period hyperbolic components of the multicorns, suggest that one should find small multicorn-like sets (based at hyperbolic components of odd periods) in the multicorns as well. These heuristics are confirmed by numerical experiments. However, it was conjectured that these ‘straightening maps’ are not homeomorphisms; a fact that we shall prove in Chapter 7 and Appendix A of this thesis.

1.2 Overview of The Main Results

The goal of this thesis is to explore the combinatorial, topological and analytic properties of the multicorns, and to investigate the ‘universality’ of the multicorns. More precisely, the principal questions that we address include the topological differences between the multicorns and their holomorphic counterparts, the *multibrot* sets [EMS14], which are the connectedness loci of unicritical holomorphic polynomials $z^d + c$, and the behavior of the ‘straightening map’ between baby multicorn-like sets (within the multicorns, and in real

cubics and other real forms [Mil92, Mil00a]) and the original multicorns.

The multicorns can be thought of as objects of intermediate complexity between one dimensional and higher dimensional parameter spaces. Combinatorially speaking, Douady's famous 'plough in the dynamical plane, and harvest in the parameter space' principle continues to stand us in good stead since our parameter space is still real two-dimensional. But since the parameter dependence of anti-polynomials is only real-analytic, one cannot typically use complex analytic techniques to study the multicorns directly. This can be circumvented by passing to the second iterate, and embedding the family $\bar{z}^d + c$ in the family $\mathcal{F}_d = \{(z^d + a)^d + b : a, b \in \mathbb{C}\}$ of holomorphic polynomials. The intersection of the real 2-dimensional slice $\{a = \bar{b}\}$ with the connectedness locus of $\mathcal{F}_d$ is precisely the multicorn $\mathcal{M}_d^*$. The situation then becomes quite similar to cubic polynomials as the maps in $\mathcal{F}_d$ have two infinite critical orbits, and one can employ some of the techniques used in studying higher dimensional parameter spaces.

As in polynomial iteration theory, the dynamics of an antiholomorphic polynomials on the plane induces a simpler dynamical system on $\mathbb{R}/\mathbb{Z}$ via its action on the dynamical rays. This is a combinatorial object that one can study with greater ease, and recover many properties of the original dynamical system. One of the key tools in this approach is the notion of orbit portraits, which was first introduced by Goldberg and Milnor in [Gol92, GM93, Mil00b] to describe the pattern of all periodic dynamical rays landing at different points of a periodic cycle for complex polynomials. The usefulness of orbit portraits stems from the fact that these combinatorial objects contain substantial information on the connection between the dynamical and the parameter planes of the maps under consideration.

Chapter 2 is devoted to the study of orbit portraits of unicritical anti-polynomials. In Section 2.1, we define orbit portraits for unicritical anti-polynomials, and note their basic properties. This is followed by examples of orbit portraits of various types. The classification Theorem 2.1.8 asserts that these are the only types of orbit portraits that can occur in our setting. The proof of this theorem involves a careful study of the characteristic angles of an orbit portrait, and this is carried out in Section 2.1.2. For even-periodic cycles, the first return map is holomorphic, and the situation is completely similar to that of holomorphic unicritical polynomials. However, when the period of a cycle is odd, the first return map is orientation-reversing, and the combinatorics of the orbit portraits associated with them turn out to be quite restricted thanks to Lemma 2.1.13, which states that at most 3 periodic dynamical rays can land at a periodic point of odd period of a unicritical anti-polynomial. In Section 2.2, we define *formal orbit portraits* as a finite collection of finite subsets of $\mathbb{Q}/\mathbb{Z}$ satisfying some properties, and we prove a realization theorem in the sense that, for every formal orbit portrait, there exists a unicritical anti-polynomial (outside the multicorns) with a periodic

cycle admitting the given formal orbit portrait. The results of Chapter 2 have been published in the paper [Muk15].

The main purpose of Chapter 3 is to reveal the structure of the boundaries of the hyperbolic components and bifurcation phenomena. When the period is even, we show that the branched covering property of the multiplier map (which is real-analytic but not holomorphic) extends to the boundary as if it were holomorphic. This implies that bifurcations from the boundary of an even period hyperbolic component occur at parabolic parameters. This is the same as the polynomial family $p_c(z) = z^d + c : c \in \mathbb{C}$, see Figure 1.4 (left). The following theorem, which is proved in Section 3.1, confirms this statement.

Theorem 1.2.1 (Bifurcations From Even Periods). *If a unicritical anti-polynomial f_c has a $2k$ -periodic cycle with multiplier $e^{2\pi i p/q}$ (with $\gcd(p, q) = 1$), then c sits on the boundary of a hyperbolic component of period $2kq$ (and is the root thereof).*

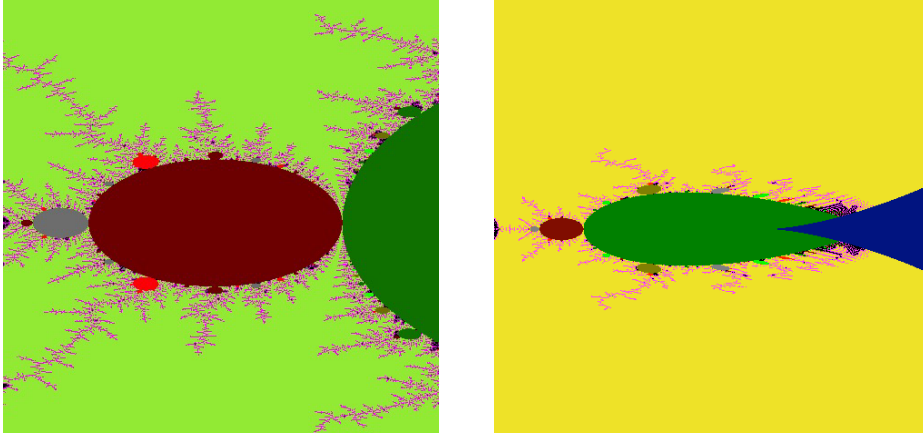


Figure 1.4: Left: A period 4 hyperbolic component of the tricorn; right: bifurcation along arcs from the period 1 component to period 2.

On the other hand, the boundary of a hyperbolic component of odd period k consists only of parabolic parameters of period k and multiplier $+1$. In Theorem 3.2.1, we show that every non-cusp odd period parabolic parameter (see Definition 1.1.16) lies in the interior of a real-analytic arc of quasi-conformally conjugate parabolic parameters. These arcs are called parabolic arcs. It had been observed numerically long ago that the boundary of every odd period hyperbolic component of $\mathcal{M}_d^*$ consists of finitely many parabolic arcs and as many cusp points. In Section 3.2 and Section 3.3, we develop the techniques and notions required to investigate the combinatorics and the topological structure of boundaries of the odd period hyperbolic components. Since we lose holomorphic parameter dependence in the parameter spaces of anti-polynomials, one needs to carry out many proofs using more combinatorial methods. The

combinatorial tools developed in Chapter 2, and certain combinatorial rigidity results turn out to be of much help here. The following theorem, which is proved in Section 3.4, can be viewed as a culmination point of these lines of ideas.

Theorem 1.2.2 (Boundary of Odd Period Components). *The boundary of every hyperbolic component of odd period is a simple closed curve consisting of exactly $d + 1$ parabolic cusp points as well as $d + 1$ parabolic arcs, each connecting two parabolic cusps.*

In Section 3.5, we utilize our work on orbit portraits and wake structures to prove discontinuity of landing points of external dynamical rays, in contrast to the situation for the Mandelbrot set. This phenomenon is reminiscent of the parameter spaces of cubic (or higher degree) polynomials.

In Section 3.6, we relate the number of hyperbolic components of a given period of the multicorns to the corresponding number for the multibrot sets (the connectedness loci of $p_c(z) = z^d + c$, denoted by $\mathcal{M}_d$). These numbers coincide for most periods, with exceptions only when the period k is twice an odd number.

Theorem 1.2.3 (Number Of Hyperbolic Components). *Denote the number of hyperbolic components of period k in $\mathcal{M}_d^*$ (respectively $\mathcal{M}_d$) by $s_{d,k}^*$ (respectively $s_{d,k}$). Then, $s_{d,k}^* = s_{d,k}$ unless k is twice an odd integer, in which case we have $s_{d,k}^* = s_{d,k} + 2s_{d,k/2}$.*

The results of Chapter 3 were proved and will be published jointly with Shizuo Nakane and Dierk Schleicher. A pre-print of the paper can be found on arXiv [MNS14].

In Chapter 4, we take a dimension-theoretic look at the persistently parabolic loci of our parameter spaces. It follows from classical works of Bowen and Ruelle [Rud87, Zin00] that the Hausdorff dimension of the Julia set depends real-analytically on the parameter within every hyperbolic component of $\mathcal{M}_d^*$. Ruelle's proof makes essential use of the fact that hyperbolic rational maps are expanding (this allows one to use the powerful machinery of thermodynamic formalism), and the Julia sets of hyperbolic rational maps move holomorphically inside every hyperbolic component. Since parabolic maps have a certain weak expansion property, and since there are real-analytic arcs of quasi-conformally conjugate parabolic parameters on the boundary of every hyperbolic component of odd period of $\mathcal{M}_d^*$, it is natural to ask whether the Hausdorff dimension of the Julia set depends real-analytically on the parameter along these parabolic arcs.

An apparent obstruction to proving real-analyticity of Hausdorff dimension of the Julia set along the parabolic arcs is that the parabolic arcs are real one-dimensional curves, and hence one cannot find a holomorphic motion of the Julia sets (of the parabolic parameters) within the family of unicritical

antiholomorphic polynomials. We circumvent this problem by constructing a strictly larger quasi-conformal deformation class of the odd period (non-cusp) parabolic parameters of the multicorns so that the deformation is no longer contained in the family of unicritical antiholomorphic polynomials, but lives in a bigger family of holomorphic polynomials. This helps us to embed the parabolic arcs (which are real one-dimensional curves) in a complex one-dimensional family of quasi-conformally conjugate parabolic maps. This proves the existence of holomorphic motion of the Julia sets under consideration (better yet, this proves structural stability of the odd period non-cusp parabolic maps along a suitable algebraic curve). This is performed in Section 4.1 by varying the critical Ecalle height over a bi-infinite strip by a quasi-conformal deformation argument.

Having the holomorphic motion of the Julia sets (of the odd period non-cusp parabolic maps) at our disposal, we can apply the results on real-analyticity of Hausdorff dimension of Julia sets of analytic families of meromorphic functions, as developed in [SU14], to our setting. The following theorem, which is proved in Section 4.2, can be naturally thought of as a version of Ruelle's theorem on the boundaries of odd period hyperbolic components:

Theorem 1.2.4 (Real-analyticity of HD along Parabolic Arcs). *Let $\mathcal{C}$ be a parabolic arc of $\mathcal{M}_d^*$ and let $c : \mathbb{R} \rightarrow \mathcal{C}, h \mapsto c(h)$ be its critical Ecalle height parametrization. Then the function*

$$\mathbb{R} \ni h \mapsto \text{HD}(J(f_{c(h)}))$$

is real-analytic.

It will transpire from the course of the proof that in good situations, the real-analyticity of Hausdorff dimension holds more generally on certain regions of the parabolic curves $\text{Per}_n(1)$ (see [Mil92] for the definition of the Per curves).

As a by-product of the q.c. deformation step, we prove that the Ecalle height parametrization of the parabolic arcs of the multicorns is non-singular at all but possibly finitely many points.

Theorem 1.2.5 (Non-singularity of Ecalle Height Parametrization). *Let $\mathcal{C}$ be a parabolic arc of odd period of $\mathcal{M}_d^*$ and $c : \mathbb{R} \rightarrow \mathcal{C}$ be its critical Ecalle height parametrization. Then, there exists a holomorphic map $\varphi : \{w = u + iv \in \mathbb{C} : |v| < \frac{1}{4}\} \rightarrow \mathbb{C}$ such that:*

1. *The map φ agrees with the map c on $\mathbb{R}$.*
2. *For all but possibly finitely many $x \in \mathbb{R}$, $c'(x) = \varphi'(x) \neq 0$.*

In particular, the critical Ecalle height parametrization of any parabolic arc of $\mathcal{M}_d^$ has a non-vanishing derivative at all but possibly finitely many points.*

It is worth mentioning that Theorem 1.2.4 adds one more item to the list of topological differences between the multicorns and their holomorphic counterparts, the multibrot sets. Clearly, the parameter dependence of the Hausdorff dimension of the Julia sets is far from regular on the boundary of the Mandelbrot set.

The results of Chapter 4 are being published, and a pre-print is available on arXiv [Muk14].

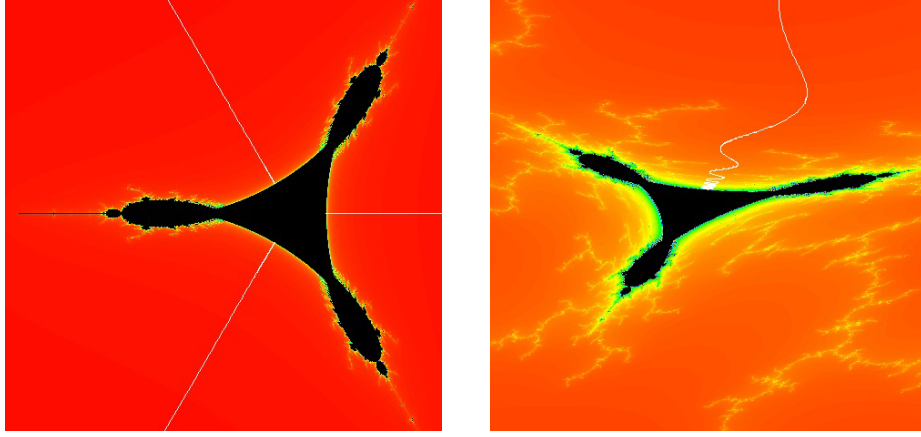


Figure 1.5: Left: Landing of the parameter rays at fixed angles on the parabolic arcs of period 1, which contain undecorated sub-arcs. Right: Non-trivial accumulation of a parameter ray at an odd-periodic angle.

Chapter 5 contains a complete description of the landing/accumulation properties of the rational parameter rays of $\mathcal{M}_d^*$ (see Theorem 5.3.2). The landing of rational parameter rays of the Mandelbrot set is related to the fact that the parabolic parameters with given combinatorics are isolated in the Mandelbrot set [GM93, Theorem C.7], [Sch00, Proposition 3.1]. This remains true for even period parabolics of the multicorns. Hence, if the accumulation set of a parameter ray $\mathcal{R}_t^d$ of $\mathcal{M}_d^*$ contains a parameter c having an even-periodic parabolic cycle, then it lands at a single point. This statement is proved in Lemma 3.6.2. But the odd periodic parabolic parameters of the multicorns are far from being isolated, there are real-analytic arcs of combinatorially equivalent parabolic parameters of any given odd period. This, at least heuristically, tells that there is no good reason for a parameter ray to land at a single point of a parabolic arc, unless it does so for some symmetry reasons. The following theorem confirms this heuristics (compare Figure 1.5).

Theorem 1.2.6 (Non-landing Parameter Rays). *The accumulation set of every parameter ray accumulating on the boundary of a hyperbolic component of odd period (except period one) of $\mathcal{M}_d^*$ contains an arc of positive length.*

The wiggling behavior is stated precisely and proved in Section 5.2. Its proof is based on the analysis of certain geometric properties of the repelling Fatou coordinates and transferring them to the parameter plane by a perturbation argument. Section 5.3 gives a complete description of which rational parameter rays of the multicorns land and which ones have this wiggling property, in terms of the combinatorics of the angle.

It is worth noting that non-trivial accumulation of some stretching rays in the parameter space of real cubic polynomials was proved by Nakane and Komori in [KN04] by different methods. It has been empirically observed that there are infinitely many small tricorn-like sets in the parameter space of real cubic polynomials. Our techniques can be naturally generalized to these small tricorn-like sets yielding the non-trivial accumulation of the stretching rays that approach the parabolic arcs on the boundaries of these small tricorn-like sets.

In Chapter 6, we will study another topological property of the multicorns. In [HS14], it was asked whether the parabolic arcs of the multicorns can contain undecorated sub-arcs. In section 6, we answer this question affirmatively (see Figure 1.5) by showing that:

Theorem 1.2.7 (Undecorated Arcs on The Boundary). *For $d \geq 2$, every period 1 parabolic arc of $\mathcal{M}_d^*$ contains an undecorated sub-arc.*

Once again, the key idea is to transfer a geometric property of the repelling Ecalle cylinder to the parameter plane using perturbation techniques. There are some interesting consequences of the previous theorem. One of them is that the centers of hyperbolic components as well as the Misiurewicz parameters (with strictly pre-periodic critical points) are not dense on the boundary of $\mathcal{M}_d^*$ (Corollary 6.2.1). This is another item in the list of the topological differences between the multicorns and the multibrot sets. The fact that the centers of hyperbolic components (or the Misiurewicz parameters) are dense on the boundaries of the multibrot sets follows by an easy application of Montel's theorem.

Equidistribution problems are of great interest in the study of the parameter spaces of polynomials (or rational maps) and a good deal of work has been done in this direction in the recent years (see [DF07, Duj14, Duj09, FG13] and the references therein). In this spirit, we deduce in Corollary 6.2.2 that the centers of the hyperbolic components of the multicorns are not equidistributed with respect to their harmonic measures. The situation is, once again, opposite to that for the multibrot sets (see [Lev90]). This essentially tells that the harmonic measures of the multicorns are rather uninteresting from a dynamical point of view. However, the question of finding a more dynamically meaningful measure (an analogue of the bifurcation measure) for the multicorns remains open.

The results of Chapter 5 and Chapter 6 are being published jointly with Hiroyuki Inou. A pre-print of the paper can be found on arXiv [IM14].

Many of the results discussed hitherto confirm that the combinatorics and topology of the multicorns greatly differ from those of their holomorphic counterparts, the multibrot sets. At the level of combinatorics, this manifests itself in the structure of orbit portraits (Theorem 2.1.8). The topological features of the multicorns have quite a few properties in common with the connectedness locus of real cubic polynomials, e.g. discontinuity of landing points of dynamical rays (Theorem 3.5.2), bifurcation along arcs (Theorem 1.1.20), existence of persistently parabolic parameters (Theorem 3.2.1), lack of local connectedness of the connectedness loci [HS14, Lav89], non-landing stretching rays ([KN04], Theorem 1.2.6), etc. These are in stark contrast with the multibrot sets.

In order to motivate and to better explain the final result of this thesis, we need to briefly review the role of polynomial-like maps in holomorphic dynamics. One of the most powerful tools in the study of dynamical systems is that of renormalization, which is a procedure of viewing the first return map of a subset of the phase space as a new (and often simpler) dynamical system. In the celebrated paper [DH85b], Douady and Hubbard developed the theory of polynomial-like maps to study renormalizations of complex polynomials, and proved the *straightening theorem* that allows one to study a sufficiently large iterate of a polynomial by associating a simpler dynamical system, namely a polynomial of smaller degree, to it. They used it to explain the existence of small homeomorphic copies of the Mandelbrot set in itself. This gives rise to the notion of ‘straightening maps’ between certain prototypical objects in parameter spaces. The fact that baby Mandelbrot sets are homeomorphic to the original one, is in some sense, a strictly ‘no interaction among critical orbits’ phenomenon. The existence of critical orbit relations for higher degree polynomials allow for much more complicated dynamical and parameter space configurations, and the corresponding straightening maps are typically not as well-behaved as in the unicritical holomorphic case.

Substantial progress in understanding the combinatorics and topology of straightening maps for higher degree polynomials has been made by Epstein (manuscript), Inou and Kiwi [IK12, Ino09] in recent years. Inou showed that the straightening map is typically discontinuous in the presence of critical orbit relations. Inou’s proof of discontinuity of the straightening map, however, makes essential use of two complex dimensional bifurcations, and can not be applied to one-parameter families. Thus, the question whether straightening maps could fail to be continuous in one-parameter families, remained open. In particular, it was conjectured that straightening maps for quadratic anti-polynomials are discontinuous, provided that the renormalization period is odd. The main purpose of Chapter 7 is to prove this conjecture for every even degree unicritical anti-polynomial family.

Theorem 1.2.8 (Discontinuity of Straightening). *Let d be even, c_0 be the center of a hyperbolic component H of odd period (other than 1) of $\mathcal{M}_d^*$, and $\mathcal{R}(c_0)$ be the corresponding c_0 -renormalization locus. Then the straightening*

map $\chi_{c_0} : \mathcal{R}(c_0) \rightarrow \mathcal{M}_d^*$ is discontinuous (at infinitely many explicit parameters).

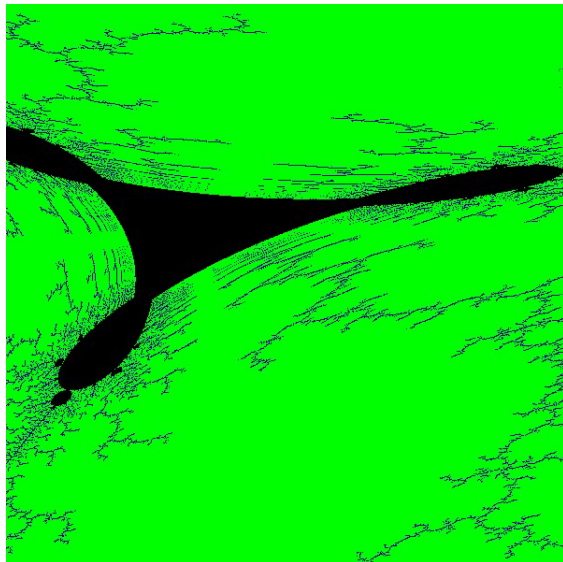


Figure 1.6: Wiggling of an umbilical cord on the root parabolic arc of a hyperbolic component of period 5 of the tricorn.

The proof of discontinuity is carried out by showing that the straightening map from a baby multicorn-like set to the original multicorn sends certain ‘wiggly’ curves to landing curves. More precisely, for even degree multicorns, there exist hyperbolic components H intersecting the real line, and their ‘umbilical cords’ land on the root parabolic arc on ∂H . In other words, such a component can be connected to the period 1 hyperbolic component by a path. However, we will prove that if H does not intersect the real line or its rotates, then no curve γ contained in $\mathcal{M}_d^* \setminus \overline{H}$ can land on the root parabolic arc on ∂H (this holds for multicorns of any degree). This means that the ‘umbilical cords’ of non-real hyperbolic components always wiggle (see Figure 1.6). Hence, for even degree multicorns, the (inverse of the) straightening map sends a piece of the real line to a ‘wiggly’ curve. The following theorem generalizes the main result of [HS14], and shows that path-connectivity fails to hold in a very strong sense for the multicorns. We should mention that this is a major topological difference from the Mandelbrot set. In fact, any two Yoccoz parameters (i.e. at most finitely renormalizable parameters) in the Mandelbrot set can be connected by an arc in the Mandelbrot set [Sch98, Theorem 5.6], [PR08].

Theorem 1.2.9 (Umbilical Cord Wiggling). *Let H be a hyperbolic component of odd period k of $\mathcal{M}_d^*$, $\mathcal{C}$ be the root arc on ∂H , and $\tilde{c}$ be the critical Ecalle*

height 0 parameter on $\mathcal{C}$. If there is a path $p : [0, \delta] \rightarrow \mathbb{C}$ with $p(0) = \tilde{c}$ and $p((0, \delta]) \subset \mathcal{M}_d^ \setminus \overline{H}$, then $\tilde{c} \in \mathbb{R} \cup \omega\mathbb{R} \cup \omega^2\mathbb{R} \cup \dots \cup \omega^d\mathbb{R}$, where $\omega = \exp(\frac{2\pi i}{d+1})$.*

The existence of non-landing umbilical cords for the multicorns was first proved by Hubbard and Schleicher [HS14] (see also [NS96]) under a strong assumption of non-renormalizability. In order to demonstrate discontinuity of the straightening map, we need to get rid of this hypothesis; i.e. we need to prove wiggling of umbilical cords for all non-real hyperbolic components. The starting point of our proof is the theory of perturbation of antiholomorphic parabolic points as developed in [HS14]. Using these perturbation techniques, they showed that the landing of an umbilical cord at $\tilde{c}$ implies that the parabolic tree of $f_{\tilde{c}}$ contains a real-analytic arc connecting two bounded Fatou components. With the assumption of non-renormalizability, one can deduce from the above statement that the entire parabolic tree is a real-analytic arc, and this implies that $f_{\tilde{c}}$ and $f_{\tilde{c}^*}$ (here, and in the sequel, z^* will stand for the complex conjugate of the complex number z) are conformally conjugate, proving that $\tilde{c}$ lies on the real line (or one of its rotates).

In the general case, we have to adopt a different strategy. From the existence of a small real-analytic arc connecting two bounded Fatou components, we show that the characteristic parabolic germs of $f_{\tilde{c}}$ and $f_{\tilde{c}^*}$ are conformally conjugate by a local biholomorphism that preserves the critical orbit tails. This is the fundamental step in our proof. Since there exists an infinite-dimensional family of conformal conjugacy classes of parabolic germs [Eca75, Vor81]; heuristically speaking, it is extremely unlikely that the parabolic germs of two conformally different polynomials would be conformally conjugate. The next step in our proof is to systematically extend the local analytic conjugacy between the parabolic germs to a conformal conjugacy between two polynomial-like restrictions. Once this is achieved, we apply a theorem of [Ino11] to conclude that some iterates of $f_{\tilde{c}}$ and $f_{\tilde{c}^*}$ are globally conjugate by a finite-to-finite holomorphic correspondence. This means that some iterates of $f_{\tilde{c}}$ and $f_{\tilde{c}^*}$ are (globally) polynomially semi-conjugate to a common polynomial. The final step is to show that $\tilde{c}$ is conformally conjugate to a real parameter, by using the theory of decompositions of polynomials with respect to composition, which is due to Ritt [Rit22] and Engstrom [Eng41].

It is worth mentioning that the proof of discontinuity of the straightening map for general polynomial families given in [Ino09] also involves proving the existence of analytically conjugate polynomial-like restrictions. The main difference is that, in higher dimensional parameter spaces, continuity of the straightening map allows one to find richer perturbations to obtain analytically conjugate polynomial-like maps. Indeed, one of the main technical steps in [Ino09] is to show (using parabolic implosion techniques) that continuity of the straightening map forces certain hybrid equivalences to preserve the moduli of multipliers of repelling periodic points of certain polynomial-like maps, and this implies that the hybrid equivalence can be promoted to an analytic

equivalence. On the other hand, the present proof employs a one-dimensional parabolic perturbation to first obtain an analytic conjugacy between parabolic germs, which is then promoted to an analytic conjugacy between polynomial-like restrictions. However, both the proofs have a common philosophy: that is to show that continuity of the straightening map would force certain hybrid equivalences to preserve some of the ‘external conformal information’.

More generally, one expects the existence of multicorn-like sets in any family of polynomials or rational maps with (at least) two critical orbits such that a pair of critical orbits are symmetric with respect to an antiholomorphic involution. Numerical evidences of this fact can be found in the recent works on the parameter spaces of certain families of rational maps, such as the family of antipode preserving cubic rationals [BBM15], Blaschke products [CFG15], etc. Although not all of our techniques can be applied to such families of rational maps, the parabolic perturbation arguments and the local consequence of umbilical cord landing (Section 7.1) do work in a general setting, and paves the way for studying analogous questions for rational maps.

In Appendix A, we will look at the tricorn-like sets in the parameter space of real cubic polynomials. The existence of small tricorn-like sets in the parameter plane of real cubic polynomials was numerically observed by Milnor [Mil92]. We discuss the landing/wiggling behavior of the umbilical cords for these tricorn-like sets and the properties of the straightening maps from these tricorn-like sets to the original tricorn. The results of this appendix are morally and technically similar to those of Chapter 7; we prove the wiggling phenomena of all non-real umbilical cords for these tricorn-like sets, and conclude that the corresponding straightening maps are discontinuous.

Appendix B is a brief survey of some dynamically defined conformal conjugacy invariants that one can associate with rational maps with interacting critical orbits. More precisely, when two or more critical orbits are attracted by an attracting or a parabolic cycle, one can study the relative positions of these orbits with respect to each other in certain uniformizing coordinates (e.g. Koenig’s coordinates, Fatou coordinates). For parabolic cycles, there is an additional invariant, namely the fixed point index. We define these invariants, and ask some questions about their mutual relations.

Appendix C deals with some local-global questions for polynomial parabolic germs. We recall some known facts about extended horn maps, and using their mapping properties, show that two conformally distinct unicritical holomorphic polynomials each having a parabolic cycle with a single petal (at each parabolic point) cannot have conformally conjugate parabolic germs. We also show that one can recover the non-cusp odd period parabolic parameters of the multicorn, up to some natural rotational and reflection symmetries, from their parabolic germs.

In Appendix D, we look at some examples of the algebraic sets $\text{Per}_n(1)$ (in various families of polynomials), and try to understand the nature of their

singularities as well the ‘dynamical’ behavior of these singular parameters. We do not prove any precise theorem here, we rather investigate some concrete examples and give heuristic explanations of singularity/non-singularity, with a view towards a more general understanding of the topology of these algebraic sets.

Chapter 2

Combinatorics: Orbit Portraits

Orbit portraits were introduced by Lisa Goldberg and John Milnor [GM93, Mil00b] as a combinatorial tool to describe the patterns of all periodic dynamical rays landing on a periodic cycle of a quadratic polynomial. This encodes information about the dynamics and the parameter spaces of these maps. In this chapter, we carry out a similar analysis for unicritical anti-polynomials.

We define orbit portraits, and state some of its basic properties in Section 2.1. The properties of the orbit portraits associated with even periodic cycles are very similar to the classical holomorphic setting. However, since the first return map of an odd periodic cycle reverses the orientation of the corresponding dynamical rays, the associated orbit portraits have quite restricted combinatorics; for instance, we show that at most 3 periodic dynamical rays can land at a periodic point of odd period of a unicritical anti-polynomial. These features are detailed in a series of lemmas in Subsection 2.1.2 culminating in the structure Theorem 2.1.8. In Section 2.2, we define *formal orbit portraits* as a finite collection of finite subsets of $\mathbb{Q}/\mathbb{Z}$ satisfying some properties, and we prove a realization theorem in the sense that, for every formal orbit portrait, there exists a unicritical anti-polynomial (outside the multicorns) with a periodic cycle admitting the given formal orbit portrait. These results serve as a combinatorial foundation for a detailed understanding of the multicorns.

The results of this chapter have been published [Muk15].

2.1 Definitions and Basic Properties

In this section, we define orbit portraits for unicritical anti-polynomials, and prove some of their basic properties.

We measure angles in the fraction of a whole turn, i.e. our angles are elements of $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$. We define for two different angles $\vartheta_1, \vartheta_2 \in \mathbb{R}/\mathbb{Z}$, the

interval $(\vartheta_1, \vartheta_2) \subset \mathbb{R}/\mathbb{Z}$ as the open connected component of $\mathbb{R}/\mathbb{Z} \setminus \{\vartheta_1, \vartheta_2\}$ that consists of the angles we traverse if we move on $\mathbb{R}/\mathbb{Z}$ in counter-clockwise direction from ϑ_1 to ϑ_2 . Finally, we denote the length of an interval $I_1 \subset \mathbb{R}/\mathbb{Z}$ by $\ell(I_1)$ such that $\ell(\mathbb{S}_1) = 1$.

Definition 2.1.1. Let $\mathcal{O} = \{z_1, z_2, \dots, z_p\}$ be a periodic cycle of a unicritical anti-polynomial f . If a dynamical ray $R_f(t)$ at a rational angle $t \in \mathbb{Q}/\mathbb{Z}$ lands at some z_i ; then for all j , the set $\mathcal{A}_j$ of the angles of all the dynamical rays landing at z_j is a non-empty finite subset of $\mathbb{Q}/\mathbb{Z}$. The collection $\{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p\}$ will be called the Orbit Portrait $\mathcal{P}(\mathcal{O})$ of the orbit $\mathcal{O}$ corresponding to the anti-polynomial f .

An orbit portrait $\mathcal{P}(\mathcal{O})$ will be called trivial if only one ray lands at each point of $\mathcal{O}$; i.e. $|\mathcal{A}_j| = 1, \forall j$. From now on, we will denote an orbit portrait simply by $\mathcal{P}$; the associated orbit will be clear from the context.

Lemma 2.1.2 (Unlinking Property). *For any orbit portrait $\mathcal{P}$ associated with a periodic orbit of a unicritical anti-polynomial, the sets $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p$ are pairwise unlinked, that is, for each $i \neq j$ the sets $\mathcal{A}_i$ and $\mathcal{A}_j$ are contained in disjoint sub-intervals of $\mathbb{R}/\mathbb{Z}$.*

Proof. This follows from the fact that two rays cannot cross each other. $\square$

Lemma 2.1.3 (Orientation Reversal). *For any anti-polynomial f , if the dynamical ray $R(t)$ at angle t lands at a point $z \in J(f)$, then the image ray $f(R(t)) = R(-dt)$ lands at the point $f(z)$. Furthermore, if three or more dynamical rays land at z , then the cyclic order of their angles around $\mathbb{R}/\mathbb{Z}$ is reversed by the action of f ; i.e. if the rays $\{R(t_1), R(t_2), \dots, R(t_v)\}$ land at z , then the cyclic order of $\{-dt_1, -dt_2, \dots, -dt_v\}$ is the opposite of that of $\{t_1, t_2, \dots, t_v\}$.*

Proof. Since the ray $R(t)$ lands at z , it must not pass through any pre-critical point of f , hence the same is true for the image ray $R(-dt)$. Therefore, the image ray is well-defined all the way to the Julia set, and continuity of f implies that it lands at $f(z)$.

For the second part, observe that f is a local orientation-reversing diffeomorphism from z_i to z_{i+1} . Hence, it reverses the cyclic order of the rays. $\square$

Lemma 2.1.4 (Finitely Many Rays). *If a dynamical ray at a rational angle lands at some point of a periodic orbit $\mathcal{O}$ of an anti-polynomial, then only finitely many rays land at each point of $\mathcal{O}$, and all these rays are periodic.*

Remark 2.1.5. *An angle $t \in \mathbb{R}/\mathbb{Z}$ (resp. a ray $R(t)$) is periodic under multiplication by $-d$ (resp. under f) if and only if $t = a/b$ (in the reduced form), for some $a, b \in \mathbb{N}$ with $\gcd(b, d) = 1$. On the other hand, t (resp. $R(t)$) is strictly pre-periodic if and only if $t = a/b$ (in the reduced form), for some $a, b \in \mathbb{N}$ with $\gcd(b, d) \neq 1$.*

Proof. Note that if f is an anti-polynomial of degree d , then $f^{\circ 2}$ is an ordinary polynomial of degree d^2 . Also, f and $f^{\circ 2}$ have the same Böttcher maps, and $R(t)$ is the same curve viewed as a dynamical ray of f or of $f^{\circ 2}$.

If a dynamical ray at a rational angle t lands at an f -periodic point, then there is also a periodic (under f) dynamical ray $R(s)$ landing at that point. Clearly, $R(s)$ is periodic under $f^{\circ 2}$ as well. But it is well-known (see [Mil00b, Lemma 2.3]) for ordinary polynomials that if a periodic ray lands at a periodic point, then all rays landing there are periodic with the same period, and hence there are finitely many of them. This proves the lemma. $\square$

In the above lemma, we did not claim that all rays landing at a periodic point have the same period under an anti-polynomial f . In fact, if a periodic point (of an anti-polynomial) has odd period, rays of different periods (under multiplication by $-d$) can indeed land there. This was first proved in [NS03], we include a different proof largely for the sake of completeness.

Lemma 2.1.6. *If a periodic orbit (of an anti-polynomial) has even length p , and if a rational dynamical ray lands at some point of the orbit, then all the rays landing at the periodic orbit have equal period, and the common ray period can be any multiple of p .*

Proof. Consider a periodic orbit $\{z_1, z_2, \dots, z_p\}$ of even period p . The first return map $f^{\circ p}$ is holomorphic (being an even iterate of an antiholomorphic map). One can now argue as in [Mil00b, Lemma 2.3] to complete the proof. $\square$

Lemma 2.1.7 (Restricted Periods). *Let f be an anti-polynomial, and z be a periodic point of odd period p of f such that at least one periodic dynamical ray lands at z . Then the period of any dynamical ray landing at z is either p or $2p$. Moreover, the number of rays of odd period p landing at z is at most 2.*

Proof. Let $\mathcal{O}$ be a periodic orbit of odd period k for an anti-polynomial f of degree d such that all rays landing at $\mathcal{O}$ are periodic. Let $\vartheta_0 \in \mathcal{A}_j$, and define $\vartheta_{n+1} := (-d)^p \vartheta_n$. We claim that $\vartheta_2 = \vartheta_0$. Otherwise, $\vartheta_0, \vartheta_1, \vartheta_2$ are three distinct angles. There can be two cases: $\vartheta_0, \vartheta_1, \vartheta_2$ lie in clockwise or in counter-clockwise order. We work with the counter-clockwise case, the other one is similar.

Since $\vartheta_0, \vartheta_1, \vartheta_2$ lie in counter-clockwise order, the orientation reversal of multiplication by $(-d)^p$ guarantees that $\vartheta_1, \vartheta_2, \vartheta_3$ lie in clockwise order; i.e. ϑ_3 lies in the component of $\mathbb{R}/\mathbb{Z} \setminus \{\vartheta_1, \vartheta_2\}$ that does not contain ϑ_0 . Continuing this process inductively, one sees that ϑ_{n+1} belongs to the component of $\mathbb{R}/\mathbb{Z} \setminus \{\vartheta_n, \vartheta_{n-1}\}$ that does not contain ϑ_0 (as shown in Figure 2.1). Therefore, $\vartheta_n \neq \vartheta_0, \forall n \in \mathbb{N}$: which contradicts the periodicity of ϑ_0 .

For the second part, suppose there are three angles $\vartheta, \vartheta', \vartheta'' \in \mathcal{A}_j$ of period p in (say) counter-clockwise order. Then multiplication by $(-d)^p$ would map them to $\{\vartheta, \vartheta', \vartheta''\}$ in clockwise order; an impossibility. Thus there can be at most two rays of period p . $\square$

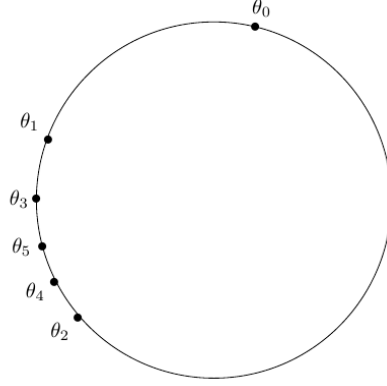


Figure 2.1: The nesting property of the orbit of ϑ_0 contradicts the periodicity.

2.1.1 Examples of Orbit Portraits

1. Even orbit period. Let $f(z) = \bar{z}^2 - 5/4$. Then f admits the orbit portrait $\{\{1/5, 4/5\}, \{2/5, 3/5\}\}$, and the angles are transitively permuted by the map $\vartheta \mapsto -2\vartheta$.

2. Odd orbit period, transitivity. Let $f(z) = \bar{z}^2 - 7/4$. This map has a parabolic 3-cycle, and all the angles of the associated orbit portrait $\{\{3/7, 4/7\}, \{2/7, 5/7\}, \{1/7, 6/7\}\}$ are permuted transitively by the dynamics.

3. Odd orbit period, non-transitivity. Let $f(z) = \bar{z}^2 - 1.77$. There is a periodic cycle of period 3, and the corresponding orbit portrait is $\{\{1/9, 8/9\}, \{2/9, 7/9\}, \{4/9, 5/9\}\}$. All the rays are fixed by the first return map; i.e. their common ray period is 3.

4. Odd orbit period, rays with different periods. For $c \approx -1.746 + 0.008i$, the anti-polynomial $f(z) = \bar{z}^2 + c$ has a 3-periodic orbit with associated orbit portrait $\{\{3/7, 4/9, 4/7\}, \{1/7, 1/9, 6/7\}, \{5/7, 7/9, 2/7\}\}$. Note that exactly two rays of period 6, and one ray of period 3 land at each point of the orbit.

Pictorial illustrations of various types of orbit portraits can be found in [NS03, Figure 5].

2.1.2 Classification of Orbit Portraits

So far all our discussions hold for general anti-polynomials of degree d . In this sub-section, we investigate the consequences of unicriticality on orbit portraits, and give a complete classification of the orbit portraits that can arise in this setting. The main theorem is the following:

Theorem 2.1.8. *Let f be a unicritical anti-polynomial of degree d , and $\mathcal{O} = \{z_1, z_2, \dots, z_p\}$ be a periodic orbit such that at least one rational dynamical*

ray lands at some z_j . Then the associated orbit portrait (which we assume to be non-trivial) $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p\}$ satisfies the following properties:

1. Each $\mathcal{A}_j$ is a finite non-empty subset of $\mathbb{Q}/\mathbb{Z}$.
2. The map $\vartheta \mapsto -d\vartheta$ maps $\mathcal{A}_j$ bijectively onto $\mathcal{A}_{j+1}$, and reverses their cyclic order.
3. For each j , $\mathcal{A}_j$ is contained in some arc of length less than $1/d$ in $\mathbb{R}/\mathbb{Z}$.
4. For every $\mathcal{A}_i$, the translated sets $\mathcal{A}_{i,j} := \mathcal{A}_i + j/d$, ($j = 0, 1, 2, \dots, d-1$) are unlinked from each other and from all other $\mathcal{A}_m$.
5. Each $\vartheta \in \mathcal{A}_j$ is periodic under $\vartheta \mapsto -d\vartheta$, and there are four possibilities for their periods:
 - (a) If p is even, then all angles in $\mathcal{P}$ have the same period rp for some $r \geq 1$.
 - (b) If p is odd, then one of the following three possibilities must be realized:
 - i. $|\mathcal{A}_j| = 2$, and both angles have period p .
 - ii. $|\mathcal{A}_j| = 2$, and both angles have period $2p$.
 - iii. $|\mathcal{A}_j| = 3$; one angle has period p , and the other two angles have period $2p$.

We divide the proof in various lemmas, the next two are essentially due to Milnor, who proved them for quadratic polynomials in [Mil00b].

Lemma 2.1.9 (The Critical Arc). *Let f be a unicritical anti-polynomial of degree d , and $\mathcal{O} = \{z_1, z_2, \dots, z_p\}$ be an orbit of period p . Let $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p\}$ be the corresponding orbit portrait. For each $j \in \{1, 2, \dots, p\}$, $\mathcal{A}_j$ is contained in some arc of length less than $1/d$ in $\mathbb{R}/\mathbb{Z}$. Thus, all but one connected component of $(\mathbb{R}/\mathbb{Z}) \setminus \mathcal{A}_j$ maps bijectively to some connected component of $(\mathbb{R}/\mathbb{Z}) \setminus \mathcal{A}_{j+1}$, and the remaining complementary arc of $(\mathbb{R}/\mathbb{Z}) \setminus \mathcal{A}_j$ covers one particular complementary arc of $\mathcal{A}_{j+1}$ d -times, and all others $(d-1)$ -times.*

Proof. Let $\vartheta \in \mathcal{A}_j$. Let β be the element of $\mathcal{A}_j$ that lies in $[\vartheta, \vartheta + 1/d)$, and is closest to $(\vartheta + 1/d)$. Similarly, let α be the member of $\mathcal{A}_j$ that lies in $(\vartheta - 1/d, \vartheta]$, and is closest to $(\vartheta - 1/d)$. Note that there is no element of $\mathcal{A}_j$ in $(\beta, \beta + 1/d]$; otherwise the orientation reversal property of multiplication by $-d$ would be violated. Similarly, $[\alpha - 1/d, \alpha)$ contains no element of $\mathcal{A}_j$. Also, the arc (α, β) must have length less than $1/d$. We will show that the entire set $\mathcal{A}_j$ is contained in the arc (α, β) of length less than $1/d$ (compare Figure 2.2).

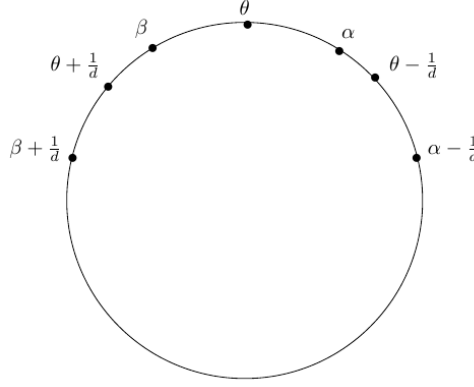


Figure 2.2: No element of $\mathcal{A}_j$ lies outside (α, β) .

If there exists some $\gamma \in \mathcal{A}_j$ lying outside (α, β) , then $\gamma \in (\beta + 1/d, \alpha - 1/d)$. Therefore, there exist at least two complementary arcs of $\mathbb{R}/\mathbb{Z} \setminus \mathcal{A}_j$ of length greater than $1/d$. Both these arcs cover the whole circle, and some other arc(s) of $\mathbb{R}/\mathbb{Z} \setminus \mathcal{A}_{j+1}$ under multiplication by $-d$. In the dynamical plane of f , the two corresponding sectors (of angular width greater than $1/d$) map to the whole plane, and some other sector(s) under the dynamics. Therefore, both these sectors contain at least one critical point of f . This contradicts the unicriticality of f .

This proves that the entire set $\mathcal{A}_j$ is contained in the arc (α, β) of length less than $1/d$. $\square$

Remark 2.1.10. *Following Milnor [Mil00b], the largest component of $(\mathbb{R}/\mathbb{Z}) \setminus \mathcal{A}_j$ (of length greater than $(1 - 1/d)$) will be called the critical arc of $\mathcal{A}_j$, and the complementary component of $\mathcal{A}_{j+1}$ that is covered d -times by the critical arc of $\mathcal{A}_j$, will be called the critical value arc of $\mathcal{A}_{j+1}$. In the dynamical plane of f , the two rays corresponding to the two endpoints of the critical arc of $\mathcal{A}_j$ along with their common landing point bound a sector containing the unique critical point of f . This sector is called a critical sector. Analogously, the sector bounded by the two rays corresponding to the two endpoints of the critical value arc of $\mathcal{A}_{j+1}$, and their common landing point contains the unique critical value of f . This sector is called a critical value sector.*

Lemma 2.1.11 (The Characteristic Arc). *Among all the complementary arcs of the various $\mathcal{A}_j$'s, there is a unique one of minimum length. It is a critical value arc for some $\mathcal{A}_j$, and is strictly contained in all other critical value arcs.*

Proof. Among all the complementary arcs of the various $\mathcal{A}_j$'s, there is clearly at least one, say (t^-, t^+) , of minimal length. This arc must be a critical value arc for some $\mathcal{A}_j$: else it would be the diffeomorphic image of some arc of $1/d$ times its length. Let (a, b) be a critical value arc for some $\mathcal{A}_k$ with $k \neq j$.

Both the critical value sectors in the dynamical plane of f contain the unique critical value (say, c) of f . Clearly, $(t^-, t^+) \cap (a, b) \neq \emptyset$. From the unlinking property of orbit portraits, it follows that (t^-, t^+) and (a, b) are strictly nested; i.e. the critical value sector (a, b) strictly contains (t^-, t^+) . $\square$

This shortest arc $\mathcal{I}_{\mathcal{P}}$ is called the *characteristic arc* of the orbit portrait, and the two angles at the ends of this arc are called the *characteristic angles*. The characteristic angles, in some sense, are crucial to the understanding of orbit portraits.

In the next lemma, we investigate the special properties of orbit portraits associated with a periodic orbit of *odd* period for a unicritical anti-polynomial in terms of their characteristic angles.

Lemma 2.1.12 (Different Periods of Characteristic Angles). *Let $\mathcal{O} = \{z_1, z_2, \dots, z_p\}$ be a periodic orbit of a unicritical anti-polynomial f with associated orbit portrait $\mathcal{P}$. If p is odd, and $|\mathcal{A}_1| \geq 3$, then one characteristic angle of $\mathcal{P}$ has period p , and the other has period $2p$.*

Proof. Without loss of generality, we can assume that the characteristic arc $\mathcal{I}_{\mathcal{P}}$ is a critical value arc of $\mathcal{A}_1$. Since $|\mathcal{A}_1| \geq 3$, $\mathcal{A}_1$ has at least three complementary components. Let $\mathcal{I}^+$ be the arc just to the right of $\mathcal{I}_{\mathcal{P}}$, and $\mathcal{I}^-$ be the one just to the left of $\mathcal{I}_{\mathcal{P}}$ (compare Figure 2.3). We can also assume that $\mathcal{I}^-$ is no shorter than $\mathcal{I}^+$; i.e. $l(\mathcal{I}^-) \geq l(\mathcal{I}^+)$. Since $\mathcal{I}^+$ is not the critical value arc of $\mathcal{A}_1$, there must exist a critical value arc $\mathcal{I}_c$ which maps diffeomorphically onto $\mathcal{I}^+$ under some iterate of multiplication by $-d$; i.e. $\mathcal{I}^+ = (-d)^m \mathcal{I}_c$, for some $m \geq 1$.

We claim that $\mathcal{I}_c = \mathcal{I}_{\mathcal{P}}$. Otherwise, $\mathcal{I}_c$ would properly contain the characteristic arc $\mathcal{I}_{\mathcal{P}}$. Since $\mathcal{I}_c$ is strictly smaller than $\mathcal{I}^+$, $\mathcal{I}_c$ cannot contain $\mathcal{I}^+$. So one end of $\mathcal{I}_c$ must lie in $\mathcal{I}^+$; but then it follows from the unlinking property that both ends of $\mathcal{I}_c$ are in $\mathcal{I}^+$. Therefore, $\mathcal{I}_c$ strictly contains $\mathcal{I}^-$. But this is impossible because $l(\mathcal{I}^-) \geq l(\mathcal{I}^+) > l(\mathcal{I}_c)$.

Therefore, $\mathcal{I}^+ = (-d)^m \mathcal{I}_{\mathcal{P}}$. Note that m must be a proper multiple of p , say $m = pq$, for some $q \geq 1$. Also let $\mathcal{I}^+ = (a, b)$, and $\mathcal{I}_{\mathcal{P}} = (b, c)$. Now we consider two cases:

Case 1 (q is even). Since $\mathcal{I}_{\mathcal{P}}$ maps to $\mathcal{I}^+$ by an orientation preserving diffeomorphism, we have: $b = d^{pq}c$, and $a = d^{pq}b$. So, $a = (d^{2p})^q c = c$ (for an odd-periodic orbit, every ray is fixed by the second return map). This contradicts the fact that $|\mathcal{A}_1| \geq 3$. So q must be odd.

Case 2 (q is odd). In this case, the arc (b, c) maps to (a, b) by an orientation reversing diffeomorphism. Hence, $b = (-d)^{pq} b$, and $a = (-d)^{pq} c$. Since b is fixed by an odd iterate of the first return map, it must have period p . On the other hand, a and c are two distinct angles such that a belongs to the orbit of c under the first return map of $\mathcal{A}_1$. Since every angle is fixed by the second return map, it follows that $\{a, c\}$ is a 2-cycle under multiplication by $(-d)^p$; hence they both have period $2p$. $\square$

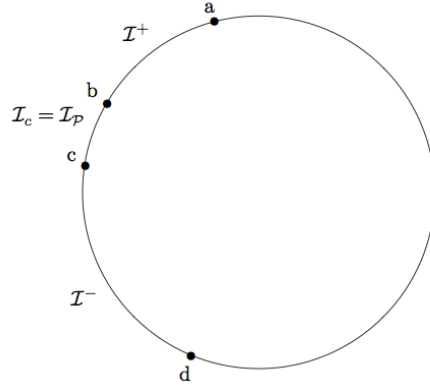


Figure 2.3: The characteristic arc $\mathcal{I}_P$ maps to the shorter adjacent arc $\mathcal{I}^+$.

The following lemma vastly limits the possibilities of orbit portraits for an odd-periodic cycle.

Lemma 2.1.13 (No More Than Three Rays). *If f is a unicritical anti-polynomial, then at most three periodic dynamical rays can land at a periodic point of odd period of f .*

Proof. Let $\mathcal{O} = \{z_1, z_2, \dots, z_p\}$ be an orbit of odd period p with associated orbit portrait $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p\}$. Suppose more than three periodic rays land at z_1 , and the characteristic arc $\mathcal{I}_P$ is the critical value arc of $\mathcal{A}_1$. Then $\mathcal{A}_1$ has at least three complementary components other than $\mathcal{I}_P$: $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$, and we can assume that $l(\mathcal{I}_1) \geq l(\mathcal{I}_2) \geq l(\mathcal{I}_3)$.

Since $\mathcal{I}_2$ is not a critical value arc, some critical value arc $\mathcal{I}_c$ must map diffeomorphically onto $\mathcal{I}_2$ under some iterate of multiplication by $-d$. Thus $\mathcal{I}_c$ is strictly smaller than $\mathcal{I}_2$. We again claim that $\mathcal{I}_c = \mathcal{I}_P$. If not, then $\mathcal{I}_c$ strictly contains $\mathcal{I}_P$. At least one end of $\mathcal{I}_c$ must lie in $\mathcal{I}_2$; else $\mathcal{I}_c$ would contain $\mathcal{I}_2$, but $l(\mathcal{I}_2) > l(\mathcal{I}_c)$. It follows from the unlinking property that both ends of $\mathcal{I}_c$ lie in $\mathcal{I}_2$. But then $\mathcal{I}_c$ would contain $\mathcal{I}_1$, which is impossible as $l(\mathcal{I}_1) \geq l(\mathcal{I}_2) > l(\mathcal{I}_c)$. Therefore, $\mathcal{I}_2 = (-d)^{m_1} \mathcal{I}_P$, with $m_1 = pr_1$ for some $r_1 \geq 1$.

Applying the same argument on $\mathcal{I}_3$, we have: $\mathcal{I}_3 = (-d)^{m_2} \mathcal{I}_P$, with $m_2 = pr_2$ for some $r_2 \geq 1$.

But from the previous lemma, the orbits of the end-points of $\mathcal{I}_P$ under multiplication by $(-d)^p$ consist of only three points: the end-points of $\mathcal{I}_P$ itself and those of one of its adjacent arcs. This contradicts the fact that both $\mathcal{I}_2$ and $\mathcal{I}_3$ are different from $\mathcal{I}_P$, and finishes the proof. $\square$

Remark 2.1.14. *An alternative proof of Lemma 2.1.13 follows from [Kiw97, Theorem 5.2]: for a unicritical anti-polynomial f , the second iterate $f^{\circ 2}$ has exactly 2 critical values, and hence, any periodic orbit portrait for $f^{\circ 2}$ can*

have at most 3 cycles of rays. However, Kiwi's theorem does not provide any information about how these dynamical rays are permuted by the anti-polynomial f . In particular, the exact periods (under f) of the characteristic angles and their relative positions are important for our applications and these are discussed in Lemma 2.1.12 and Lemma 2.2.6.

We are now in a position to prove the main theorem of this section.

Proof of Theorem 2.1.8. Property (4) simply states the fact that if two periodic rays $R_c(\vartheta)$ and $R_c(\vartheta')$ land together at some periodic point z , then the rays $R_c(\vartheta + j/d)$ and $R_c(\vartheta' + j/d)$ land together at a pre-periodic point z' with $f(z') = f(z)$, since the Julia set of f has a d -fold rotation symmetry. The rest of the properties follow from the previous lemmas. $\square$

2.2 A Realization Theorem

Definition 2.2.1. A finite collection $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p\}$ of subsets of $\mathbb{R}/\mathbb{Z}$ satisfying the five properties of Theorem 2.1.8 is called a formal orbit portrait. The property (4) of Theorem 2.1.8 implies that each $\mathcal{A}_j$ has a complementary arc of length greater than $(1 - 1/d)$ (which we call the critical arc of $\mathcal{A}_j$) that, under multiplication by $-d$ covers exactly one complementary arc of $\mathcal{A}_{j+1}$ d -times (which we call the critical value arc of $\mathcal{A}_{j+1}$), and the others $(d - 1)$ -times.

The principal goal of this section is to prove the following realization theorem.

Theorem 2.2.2. Let $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p\}$ be a formal orbit portrait. Then there exists some $c \in \mathbb{C} \setminus \mathcal{M}_d^*$, such that $f(z) = \bar{z}^d + c$ has a repelling periodic orbit with associated orbit portrait $\mathcal{P}$.

We borrow many techniques from the proof of the realization of formal orbit portraits for quadratic polynomials given in [Mil00b]. The antiholomorphic case is slightly more difficult than the holomorphic one, and the proof will be divided into several cases.

The next lemma is a combinatorial version of Lemma 2.1.11, and this is where condition (4) of the definition of formal orbit portraits comes in.

Lemma 2.2.3. Let $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p\}$ be a formal orbit portrait. Among all the complementary arcs of the various $\mathcal{A}_j$'s, there is a unique one of minimum length. It is a critical value arc for some $\mathcal{A}_j$, and is strictly contained in all other critical value arcs.

Proof. Among all the complementary arcs of the various $\mathcal{A}_j$'s, there is clearly at least one, say $\mathcal{I} = (t^-, t^+)$, of minimal length l . This arc must be a critical value arc of some $\mathcal{A}_j$, else it would be the diffeomorphic image of some arc

of $1/d$ times its length. Let $\mathcal{I}' = (a, b)$ be the critical arc of $\mathcal{A}_{j-1}$ having length $\{(d-1) + l\}/d$ so that its image, under multiplication by $-d$, covers (t^-, t^+) d -times, and the rest of the circle exactly $(d-1)$ -times. (t^-, t^+) has d pre-images $-\mathcal{I}/d, (-\mathcal{I}/d + 1/d), (-\mathcal{I}/d + 2/d), \dots, (-\mathcal{I}/d + (d-1)/d)$ (as shown in Figure 2.4); each of them is contained in (a, b) , and has length l/d . By our minimality assumption, (t^-, t^+) contains no angle of $\mathcal{P}$, and hence neither do its d pre-images. Label the d connected components of $\mathbb{R}/\mathbb{Z} \setminus \bigcup_{r=0}^{d-1} (-\mathcal{I}/d + r/d)$ as $C_1, C_2, \dots, C_d$ with $C_1 = [b, a]$.

Clearly, $\mathcal{A}_{j-1}$ is contained in C_1 , and the two end-points a and b of C_1 belong to $\mathcal{A}_{j-1}$. Also, $C_{i+1} = C_1 + i/d$ for $0 \leq i \leq d-1$. Therefore, $\mathcal{A}_{j-1} + i/d$ is contained in C_{i+1} with the end-points of C_{i+1} belonging to $\mathcal{A}_{j-1} + i/d$. By condition (4) of the definition of formal orbit portraits, each $\mathcal{A}_{j-1} + i/d$ (for fixed j , and varying i) is unlinked from $\mathcal{A}_k$, for $k \neq j-1$. This implies that for any $k \neq j-1$, $\exists! r_k \in \{1, 2, \dots, d\}$ such that $\mathcal{A}_k$ is contained in $\text{int}(C_{r_k})$. Hence, all the non-critical arcs of $\mathcal{A}_k$ would be contained in the interior of C_{r_k} . Thus all the non-critical value arcs of $\mathcal{A}_{k+1}$ ($k+1 \neq j$) are contained in $\mathbb{R}/\mathbb{Z} \setminus [t^-, t^+]$. Hence the critical value arc of any $\mathcal{A}_m$ ($m \neq j$) strictly contains $\mathcal{I} = (t^-, t^+)$. The uniqueness follows. $\square$

We will also need an antiholomorphic analogue of a classical result about unicritical polynomials (see [Mil00b, Lemma 2.7], [Sch00, Lemma 2.4]).

Lemma 2.2.4. *For a formal antiholomorphic orbit portrait $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p\}$ with even p , multiplication by $-d$ either permutes all the angles of $\mathcal{P}$ transitively or $|\mathcal{A}_j| = 2 \forall j$, and the first return map of $\mathcal{A}_j$ fixes each angle.*

Proof. We assume that the cardinality of each $\mathcal{A}_j$ is at least three, and we will show that multiplication by $-d$ permutes all the angles of $\mathcal{P}$. We can also assume that the characteristic arc $\mathcal{I}_{\mathcal{P}}$ is a critical value arc of $\mathcal{A}_1$. Since $|\mathcal{A}_1| \geq 3$, $\mathcal{A}_1$ has at least three complementary components. Let $\mathcal{I}^+$ be the arc just to the right of $\mathcal{I}_{\mathcal{P}}$, and $\mathcal{I}^-$ be the one just to the left of $\mathcal{I}_{\mathcal{P}}$. Let $\mathcal{I}^-$ be longer than $\mathcal{I}^+$; i.e. $l(\mathcal{I}^-) \geq l(\mathcal{I}^+)$. Since $\mathcal{I}^+$ is not the critical value arc of $\mathcal{A}_1$, there must exist a critical value arc $\mathcal{I}_c$ which maps diffeomorphically onto $\mathcal{I}^+$ under some iterate of multiplication by $-d$; i.e. $\mathcal{I}^+ = (-d)^m \mathcal{I}_c$, for some $m \geq 1$. Arguing as in Lemma 2.1.12, we see that $\mathcal{I}_c = \mathcal{I}_{\mathcal{P}}$.

Therefore, $\mathcal{I}^+ = (-d)^m \mathcal{I}_{\mathcal{P}}$. Note that m must be multiple of p , thus m is even. Also let $\mathcal{I}^+ = (a, b)$, and $\mathcal{I}_{\mathcal{P}} = (b, c)$. Since $\mathcal{I}_{\mathcal{P}}$ maps to $\mathcal{I}^+$ by an orientation preserving diffeomorphism, we have: $b = d^m c$, and $a = d^m b$. Multiplication by d^m is an orientation preserving map, and it sends $\mathcal{A}_1$ bijectively onto itself such that the point b is mapped to an adjacent point a . It follows that multiplication by d^m acts transitively on $\mathcal{A}_1$. Hence multiplication by $-d$ permutes all the angles of $\mathcal{P}$ transitively. $\square$

The next lemma is the key to the proof of the realization theorem, and gives a necessary condition for the dynamical rays (of a unicritical anti-polynomial) at the characteristic angles of a formal orbit portrait to land at a common point.

Lemma 2.2.5 (Outside The Multicorns). *Let $\mathcal{P}$ be a formal orbit portrait, and (t^-, t^+) be its characteristic arc. For some $c \notin \mathcal{M}_d^*$, the two dynamical rays $R_c(t^-)$ and $R_c(t^+)$ land at the same point of $J(f_c)$ if the external angle $t(c) \in (t^-, t^+)$.*

Proof. We retain the terminology of Lemma 2.2.3. If $c \notin \mathcal{M}_d^*$, all the periodic points of $f_c = \bar{z}^d + c$ are repelling, and the Julia set is a cantor set.

Let the external angle of c in the parameter plane be $t(c)$. Label the connected components of $\mathbb{R}/\mathbb{Z} \setminus \{-t(c)/d, -t(c)/d + 1/d, \dots, -t(c)/d + (d-1)/d\}$ counter-clockwise as $L_0, L_1, \dots, L_{d-1}$ such that the component containing the angle 0 gets label L_0 . The $t(c)$ -itinerary of an angle $\vartheta \in \mathbb{R}/\mathbb{Z}$ is defined as a sequence $(a_n)_{n \geq 0}$ in $\{0, 1, \dots, d-1\}^{\mathbb{N}}$ such that $a_n = i$ if $(-d)^n \vartheta \in L_i$. All but countably many ϑ 's (the ones which are not the iterated pre-images of $t(c)$ under multiplication by $-d$) have a well-defined $t(c)$ -itinerary.

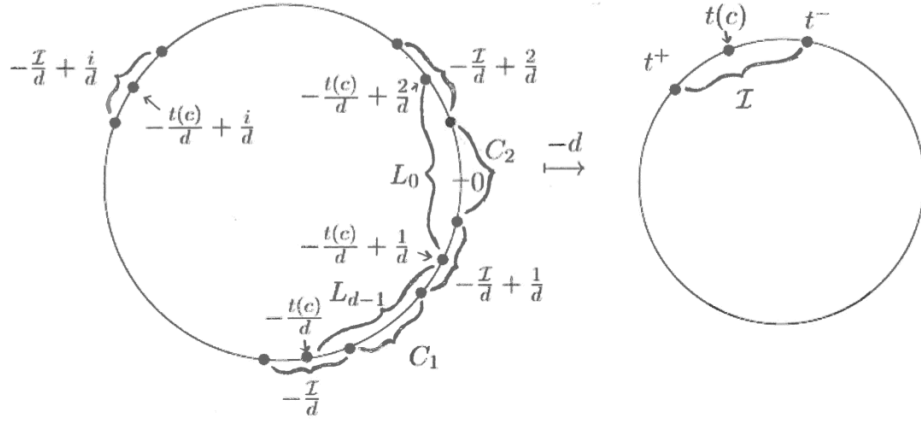


Figure 2.4: The d pre-images of $\mathcal{I} = (t^-, t^+)$ under multiplication by $-d$ and the complementary arcs C_i 's are labelled on the circle. Also the d pre-images of $t(c) \in (t^-, t^+)$ are marked, and the components L_i 's of $\mathbb{R}/\mathbb{Z} \setminus \{-t(c)/d, -t(c)/d + 1/d, \dots, -t(c)/d + (d-1)/d\}$ are labelled. Each C_i is contained in some L_j .

Similarly, in the dynamical plane of f_c , the d dynamical rays $R_c(-t(c)/d), R_c(-t(c)/d + 1/d), \dots, R_c(-t(c)/d + (d-1)/d)$ land at the critical point 0, and cut the dynamical plane into d sectors. Label these sectors counter-clockwise

as $L'_0, L'_1, \dots, L'_{d-1}$ such that the component containing the dynamical ray $R_c(0)$ at angle 0 gets label L'_0 . Any point $z \in J(f_c)$ has an associated symbol sequence $(a_n)_{n \geq 0}$ in $\{0, 1, \dots, d-1\}^{\mathbb{N}}$ such that $a_n = i$ if $f_c^n(z) \in L'_i$. Clearly, a dynamical ray $R_c(\vartheta)$ at angle ϑ lands at z if and only if the $t(c)$ -itinerary of ϑ coincides with the symbol sequence of z defined above.

If $t(c) \in \mathcal{I} = (t^-, t^+)$, the d angles $\{-t(c)/d, -t(c)/d + 1/d, \dots, -t(c)/d + (d-1)/d\}$ lie in the d intervals $-\mathcal{I}/d, (-\mathcal{I}/d + 1/d), \dots, (-\mathcal{I}/d + (d-1)/d)$ respectively, and no element of $\mathcal{P}$ belongs to $\bigcup_{j=0}^{d-1} (-\mathcal{I}/d + j/d)$. First note that the rays $R_c(t^-)$ and $R_c(t^+)$ indeed land as $t(c) \notin$ the finite sets $\{t^\pm, -dt^\pm, (-d)^2 t^\pm, \dots\}$. Each $\mathcal{A}_j$ is contained in a unique C_r . Therefore, for each $n \geq 0$, the angles $(-d)^n t^-$ and $(-d)^n t^+$ belong to the same L_i . So t^- and t^+ have the same $t(c)$ -itinerary; which implies that the two dynamical rays $R_c(t^-)$ and $R_c(t^+)$ land at the same point of $J(f_c)$. $\square$

In the next lemma, we take a closer look at orbit portraits associated with odd-periodic cycles such that exactly three rays land at each point of the cycle. It follows from Lemma 2.1.12 that two of these angles must have period $2p$, and the other one has period p such that the two characteristic angles have different periods.

Lemma 2.2.6. *Let $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p\}$ be a formal orbit portrait with p odd, and $|\mathcal{A}_1| = 3$. Assume further that $\mathcal{A}_1 = \{t^-, t, t^+\}$ with t^- and t^+ of period $2k$, and t of period k such that the characteristic angles are $\{t^-, t\}$. Then,*

1. t belongs to the shorter complementary component of $\mathbb{R}/\mathbb{Z} \setminus \{t^-, t^+\}$,
2. $\ell(\text{the shorter complementary component of } \mathbb{R}/\mathbb{Z} \setminus \{t^-, t^+\}) = (1 + d^p) \cdot \ell(\text{the characteristic arc of } \mathcal{P})$.

Proof. Let $\mathcal{I}_{\mathcal{P}}$ be the characteristic arc of $\mathcal{P}$ with end-points t^-, t , and the other two complementary components of $\mathcal{A}_1$ be $\mathcal{I}_1$ and $\mathcal{I}_2$. Without loss of generality, we can assume that $\mathcal{I}_2$ is a critical arc. So, $\ell(\mathcal{I}_2) > \ell(\mathcal{I}_1) > \ell(\mathcal{I}_{\mathcal{P}})$. An argument similar to that in Lemma 2.1.13 shows that $\mathcal{I}_{\mathcal{P}}$ must map diffeomorphically onto $\mathcal{I}_1$ under some iterate of multiplication by $-d$; i.e. $\mathcal{I}_1 = (-d)^{pq} \mathcal{I}_{\mathcal{P}}$, for some $q \in \mathbb{N}$. Since one end-point (namely, t) of $\mathcal{I}_{\mathcal{P}}$ has period p , it follows that t is an end-point of $\mathcal{I}_1$ as well, and $q = 1$. Hence, the end-points of the critical arc $\mathcal{I}_2$ are t^- and t^+ . This proves that t belongs to the shorter complementary component of $\mathbb{R}/\mathbb{Z} \setminus \{t^-, t^+\}$, and $\ell(\mathcal{I}_1) = d^p \cdot \ell(\mathcal{I}_{\mathcal{P}})$. Since the shorter complementary component of $\mathbb{R}/\mathbb{Z} \setminus \{t^-, t^+\} = \mathcal{I}_{\mathcal{P}} \cup \mathcal{I}_1$, part (ii) follows. $\square$

Proof Of Theorem 2.2.2. Let $\mathcal{P}$ be a formal antiholomorphic orbit portrait. We consider the following cases.

Case 1 (p is even).

First let us assume that $|\mathcal{A}_j| = 2$, and the first return map (multiplication by $(-d)^p = d^p$) of $\mathcal{A}_j$ fixes each angle. Let $\mathcal{I}_{\mathcal{P}} = (t^-, t^+)$ be the characteristic arc of the formal orbit portrait $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p\}$ such that $\{t^-, t^+\} \subset \mathcal{A}_1$. Choose c outside $\mathcal{M}_d^*$ with $t(c) \in (t^-, t^+)$. Then, the two dynamical rays $R_c(t^-)$ and $R_c(t^+)$ land at the same point $z \in J(f_c)$. Clearly, z is a periodic point of period $p'|p$. Let $\mathcal{P}' = \{\mathcal{A}'_1, \mathcal{A}'_2, \dots, \mathcal{A}'_{p'}\}$ be the orbit portrait associated with $\mathcal{O}(z)$ such that $\mathcal{A}'_1$ is the set of angles of the rays landing at z . Since the two elements t^- and t^+ of $\mathcal{A}'_1$ are not in the same cycle under multiplication by $-d$, Lemma 2.2.4 implies that exactly two rays land at z , and $p = p'$. Therefore, $\mathcal{P}' = \mathcal{P}$.

On the other hand, if multiplication by $-d$ acts transitively on $\mathcal{P}$, there exists $l \in \mathbb{N}$ such that $d^{lp}t^- = t^+$. Let r' rays land at each point of $\mathcal{O}(z)$ as above (z is the common landing point of $R_c(t^-)$ and $R_c(t^+)$, where c lies outside $\mathcal{M}_d^*$ with $t(c) \in (t^-, t^+)$) so that the associated orbit portrait $\mathcal{P}'$ contains a total of $r'p'$ rays, where p' is the period of z . Lemma 2.2.4 implies that multiplication by $-d$ acts transitively on $\mathcal{P}'$, and the common period of all the angles in $\mathcal{P}'$ is $r'p'$. Since t^- and t^+ are adjacent angles in $\mathcal{A}_1$, it easily follows that multiplication by d^{lp} acts transitively on $\mathcal{A}_1$. Therefore, all the angles in $\mathcal{A}_1$ land at z ; i.e. $r' \geq r$. Since $r'p' = rp$, we have $p' \leq p$. If p' was strictly smaller than p , both the sets $\mathcal{A}_1$ and $\mathcal{A}_{1+p'}$ would be contained in $\mathcal{A}'_1$; hence multiplication by d^p would map these two sets onto themselves preserving their cyclic order. This forces multiplication by d^p to be the identity map on $\mathcal{A}_1$: a contradiction to the transitivity assumption. Thus, $p' = p$, and $r' = r$. Therefore, $\mathcal{A}_1 = \mathcal{A}'_1$, and $\mathcal{P}' = \mathcal{P}$.

Case 2 (p is odd, each $\mathcal{A}_j$ consists of exactly two angles of period p).

Let $\mathcal{I}_{\mathcal{P}} = (t^-, t^+)$ be the characteristic arc of the formal orbit portrait $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p\}$ such that $\mathcal{A}_1 = \{t^-, t^+\}$. Choose c outside $\mathcal{M}_d^*$ with $t(c) \in (t^-, t^+)$. Then, the two dynamical rays $R_c(t^-)$ and $R_c(t^+)$ (of period k) land at the same point $z \in J(f_c)$. Clearly, z is a periodic point of period $p'|p$. Since z has odd period p' , the periods of rays landing there can be either p' or $2p'$. It follows that $p = p'$. Also, by Theorem 2.1.8, if two rays of period p land at a periodic point of odd period p , then these are the only rays landing there. Therefore, if $\mathcal{P}' = \{\mathcal{A}'_1, \mathcal{A}'_2, \dots, \mathcal{A}'_{p'}\}$ is the associated orbit portrait of $\mathcal{O}(z)$ such that $\mathcal{A}'_1$ is the set of angles of the rays landing at z , then $\mathcal{A}_1 = \mathcal{A}'_1$. Hence, $\mathcal{P}' = \mathcal{P}$.

Case 3 (p is odd, each $\mathcal{A}_j$ consists of exactly two angles of period $2p$ and one angle of period p).

Let $\mathcal{I}_{\mathcal{P}} = (t^-, t^+)$ be the characteristic arc of the formal orbit portrait $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p\}$ such that $\{t^-, t^+\} \subset \mathcal{A}_1$. By Lemma 2.1.12, one char-

acteristic angle has period p , and the other has period $2p$. Choose c outside the $\mathcal{M}_d^*$ with $t(c) \in (t^-, t^+)$. Then, the two dynamical rays $R_c(t^-)$ and $R_c(t^+)$ land at the same point $z \in J(f_c)$. Clearly, z is a periodic point of period $p'|p$. Since z has odd period p' , the periods of rays landing there can be either p' or $2p'$. It follows that $p = p'$. Let $\mathcal{P}' = \{\mathcal{A}'_1, \mathcal{A}'_2, \dots, \mathcal{A}'_p\}$ be the associated orbit portrait of $\mathcal{O}(z)$ such that $\mathcal{A}'_1$ is the set of angles of the rays landing at z . By Theorem 2.1.8, if one ray of period p and another ray of period $2p$ land at a periodic point of odd period p , then exactly three rays land there. Clearly, $\mathcal{A}_1 = \mathcal{A}'_1$. Hence, $\mathcal{P}' = \mathcal{P}$.

Case 4 (p is odd, each $\mathcal{A}_j$ consists of exactly two angles of period $2p$).

As above, let $\mathcal{I}_{\mathcal{P}} = (t^-, t^+)$ be the characteristic arc of the formal orbit portrait $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p\}$ such that $\mathcal{A}_1 = \{t^-, t^+\}$. Choose c outside $\mathcal{M}_d^*$ with $t(c) \in (t^-, t^+)$. Then, the two dynamical rays $R_c(t^-)$ and $R_c(t^+)$ (of period $2p$) land at the same point $z \in J(f_c)$. Clearly, z is a periodic point of period some p' . As $R_c(t^+)$ lands at z , $R_c(t^-) = f^{\circ p}(R_c(t^+))$ must land at $f^{\circ p}(z)$. So, $f^{\circ p}(z) = z$. Hence, $p'|p$. Since z has odd period p' , the periods of rays landing there can be either p' or $2p'$. It follows that $p = p'$. Let $\mathcal{P}' = \{\mathcal{A}'_1, \mathcal{A}'_2, \dots, \mathcal{A}'_p\}$ be the associated orbit portrait of $\mathcal{O}(z)$ with $\mathcal{A}'_1$ being the set of angles of the rays landing at z . By Theorem 2.1.8, if two rays of period $2p$ land at a periodic point of odd period p , then either these are the only rays landing there or there can be at most one further ray of period p landing there. In the first case, $\mathcal{A}_1 = \mathcal{A}'_1$, and thus, $\mathcal{P}' = \mathcal{P}$.

In the second case, we must work a little harder. Let $\mathcal{I}_{\mathcal{P}'}$ be the characteristic arc of the orbit portrait $\mathcal{P}'$ associated with the orbit $\mathcal{O}(z)$. Since exactly three rays (two of period $2p$ and one of period p) land at z , Lemma 2.1.12 tells that one characteristic angle has period p and the other has period $2p$. We claim that the characteristic angles of $\mathcal{P}'$ belong to $\mathcal{A}'_1$. Let us assume that this is false, and we will establish a contradiction. Note that $t(c) \in \mathcal{I}_{\mathcal{P}'}$ ($\mathcal{I}_{\mathcal{P}'}$ is the characteristic arc of an actual orbit portrait for f_c , hence it must be a critical value arc, and thus contains the external angle of the critical value c). So $\mathcal{I}_{\mathcal{P}'}$ and $\mathcal{I}_{\mathcal{P}}$ intersect, and hence must be strictly nested (unlinking property); in particular, $\mathcal{I}_{\mathcal{P}}$ strictly contains $\mathcal{I}_{\mathcal{P}'}$ (since $\mathcal{I}_{\mathcal{P}'}$ is the characteristic arc of $\mathcal{P}'$). But one end of $\mathcal{I}_{\mathcal{P}'}$ is an angle of period $2p$ which is already contained in $\mathcal{P}$, thus the characteristic arc $\mathcal{I}_{\mathcal{P}}$ of the formal orbit portrait $\mathcal{P}$ contains an element of $\mathcal{P}'$: a contradiction. Hence, the characteristic angles of $\mathcal{P}'$ belong to $\mathcal{A}'_1$.

Let $\mathcal{A}'_1 = \{t^-, t, t^+\}$, where the characteristic angles are either $\{t^-, t\}$ or $\{t, t^+\}$. Since $t(c) \in \mathcal{I}_{\mathcal{P}'} \cap \mathcal{I}_{\mathcal{P}}$, we conclude that $t \in (t^-, t^+)$. To fix our ideas, let us assume that $\mathcal{I}_{\mathcal{P}'} = (t^-, t)$ (see Figure 2.5). Since $\mathcal{P}'$ is an orbit portrait satisfying the condition of Case (3), and (t^-, t) is its characteristic arc, it follows from above that for any $c' \notin \mathcal{M}_d^*$ with $t(c') \in (t^-, t)$, $f_{c'}$ admits the orbit portrait $\mathcal{P}'$.

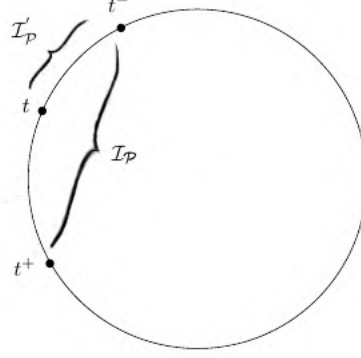


Figure 2.5: The characteristic arc $\mathcal{I}_{P'}$ of the actual orbit portrait $\mathcal{P}'$ is contained in the characteristic arc $\mathcal{I}_P$ of the formal orbit portrait $\mathcal{P}$, and they share an endpoint, say t^- , of period $2p$.

Finally, to find an anti-polynomial admitting the orbit portrait $\mathcal{P}$, choose $\hat{c} \notin \mathcal{M}_d^*$ with $t(\hat{c}) = t$. Since $t(\hat{c}) \in (t^-, t^+)$, the characteristic arc of the formal orbit portrait $\mathcal{P}$, it is now routine to check that the two dynamical rays $R_{\hat{c}}(t^-)$ and $R_{\hat{c}}(t^+)$ (of period $2p$) land at the same point $\hat{z} \in J(f_{\hat{c}})$ such that $\hat{z}$ has period p . Let $\hat{\mathcal{P}} = \{\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2, \dots, \hat{\mathcal{A}}_p\}$ be the orbit portrait associated with the periodic orbit $\mathcal{O}(\hat{z})$ with $t^-, t^+ \in \hat{\mathcal{A}}_1$. We will show that $\hat{\mathcal{A}}_1 = \{t^-, t^+\}$, which will complete the proof.

First note that the dynamical ray $R_{\hat{c}}(t)$ cannot land because the assumption $t(\hat{c}) = t$ forces the ray to bifurcate. If some other dynamical ray $R_{\hat{c}}(\tilde{t})$ ($\tilde{t} \neq t^-, t^+$) lands at $\hat{z}$, then the angle $\tilde{t}$ must have period p under multiplication by $-d$. Since $\hat{z}$ is a repelling periodic point for $f_{\hat{c}}$, and the dynamical rays at angles $\tilde{t}, t^-, t^+$ land at it, $\hat{z}$ can be real-analytically followed as a repelling periodic point of odd period p , and the corresponding dynamical rays at the same angles would continue to land there under small perturbation of $\hat{c}$. But if we choose $c' \approx \hat{c}$ so that $t(c') \in (t^-, t)$, then in the dynamical plane of $f_{c'}$, the four rays at angles $\tilde{t}, t^-, t^+$ and t would land at a common repelling periodic point of odd period: a contradiction to Lemma 2.1.13. We conclude that $\hat{\mathcal{A}}_1 = \{t^-, t^+\}$, and hence $\hat{\mathcal{P}} = \mathcal{P}$. $\square$

Remark 2.2.7. *a) In case (4) of the proof of Theorem 2.2.2, we proved the realization of the orbit portrait $\mathcal{P}$ for parameters on a certain parameter ray. One can, with a bit more effort, make the following stronger statement: there is an open subset S of $\mathbb{R}/\mathbb{Z}$ such that every parameter outside $\mathcal{M}_d^*$ having external angle in S admits the orbit portrait $\mathcal{P}$. Indeed, if $\mathcal{I}_P = (t^-, t^+)$ is the characteristic arc of the formal orbit portrait $\mathcal{P}$; then it is not hard to check that there are at most two angles $t_1, t_2 \in (t^-, t^+)$ of period p (with $t_1 < t_2$,*

say) such that the three dynamical rays $R_c(t_i)$ ($i = 1$ or 2), $R_c(t^-)$ and $R_c(t^+)$ can possibly land at a common point. Also, this can happen precisely when the external angle $t(c)$ of c lies in the intervals (t^-, t_1) and (t_2, t^+) respectively. It follows that $S := (t_1, t_2)$ satisfies the required property.

b) A unicritical anti-polynomial f_c ($c \notin \mathcal{M}_d^*$) can admit an orbit portrait $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p\}$ only if $t(c) \in (t^-, t^+)$, where (t^-, t^+) is the characteristic arc of $\mathcal{P}$. Indeed, the characteristic arc must be a critical value arc for some $\mathcal{A}_j$, and in the dynamical plane, the corresponding critical value sector bounded by the two rays $R_c(t^-)$ and $R_c(t^+)$ together with their common landing point contains the critical value c . Therefore the external angle $t(c)$ of c will lie in the interval (t^-, t^+) .

Chapter 3

Bifurcation Phenomena

In this chapter, we lay the foundation of our study of the topological structure of the multicorns. Our primary goal is to unravel the topological and combinatorial properties of the boundaries of the hyperbolic components, and to elucidate their differences from the holomorphic setting.

This chapter is organized in the following way. We will prove Theorem 1.2.1 revealing the structure of bifurcations from even period hyperbolic components, and collect some basic properties of bifurcations from odd period components in Section 3.1. We will also prove finiteness of parameters with certain given properties, which will be useful in the subsequent sections. The existence of parabolic arcs on the boundaries of odd period hyperbolic components, and the combinatorial properties of the dynamics on such arcs will be discussed in Sections 3.2 and Section 3.3. With this preparation, and with a combinatorial rigidity result (Lemma 3.4.3) at our disposal, we will prove Theorem 1.2.2 in Section 3.4. This gives a complete description of the boundaries of the odd period hyperbolic components of the multicorns. Section 3.5 exhibits a discontinuity phenomenon of the landing points of dynamical rays, and Section 3.6 counts the number of hyperbolic components of a given period in $\mathcal{M}_d^*$.

The results of this chapter will appear in a joint paper (which has been accepted for publication) with Dierk Schleicher and Shizuo Nakane. A preprint of the paper can be found on arXiv [MNS14].

3.1 Basic Properties

The parameter dependence of anti-polynomials is only real-analytic (and not holomorphic), and hence, unlike in the holomorphic case, one cannot use the open mapping principle of the multiplier map to discuss the bifurcation structure of hyperbolic components. The following theorem can be considered as a weak replacement, in certain cases, for the open mapping principle of the multiplier; recall that this is false in the parametrization $z \mapsto \mu\bar{z} + \bar{z}^2$!

Theorem 3.1.1 (Indifferent Parameters on Boundary). *If $f_{c_0}(z) = \bar{z}^d + c_0$ has an indifferent periodic point of period k , then every neighborhood of c_0 contains parameters with attracting periodic points of period k , so the parameter c_0 is on the boundary of a hyperbolic component of period k of the multicorn $\mathcal{M}_d^*$.*

Moreover, every neighborhood of c_0 contains parameters for which all period k orbits are repelling.

Proof. The idea of the proof is the same as the one for holomorphic polynomials by Douady [Dou83, III.1], so we give only a sketch. It is easier to prove this result in the family $q_\lambda: z \mapsto \lambda(1 + \bar{z}/d)^d$ as described in the introduction. The map f_{c_0} is conformally conjugate to q_{λ_0} with $\lambda_0 = d\bar{c}_0^d/c_0$.

First we restrict q_{λ_0} to an anti-polynomial-like map of degree d and perturb it slightly so that the k -cycle becomes attracting (or repelling, so that all the other k -periodic orbits remain repelling); this requires a slight adjustment of the domain of the perturbed anti-polynomial-like map. The straightening theorem supplies an anti-polynomial of equal degree with a single critical point in $\mathbb{C}$ of maximal multiplicity, so it is conformally conjugate to a map $z \mapsto \lambda'(1 + \bar{z}/d)^d$ for a unique $\lambda' \in \mathbb{C}$. The Beltrami differential in the straightening theorem can be chosen to be arbitrarily close to zero when the perturbation is small enough, so λ' can be chosen arbitrarily close to λ . This shows the result in the family q_λ , and it follows in the original family because the mapping $\lambda = d\bar{c}^d/c$ is a covering map of degree $d + 1$. $\square$

There is no analogous result to the preceding theorem for the family $P_\mu: z \mapsto \mu\bar{z} + \bar{z}^2$ because the map $\mu \mapsto \lambda(\mu)$ is not a covering map: it fails to be locally injective whenever $|\mu| = 1$.

Now we prove Theorem 1.2.1 which shows the existence of bifurcations at parabolic parameters with even periodic parabolic orbits. This is subtler to prove than in the holomorphic case: if f_{c_0} has an indifferent orbit of period k and multiplier $\mu = e^{2\pi ip/q}$ with p/q in lowest terms and $q \geq 2$, then in the holomorphic case it simply follows from the open mapping principle that c_0 is on the boundary of a hyperbolic component of period kq . In the antiholomorphic case, there is no open mapping principle, and the statement is not obvious. Our argument is based on similar ideas as for the proof of Theorem 3.1.1.

Proof of Theorem 1.2.1. Let $f_{c_0}(z) = \bar{z}^d + c_0$ be the given antiholomorphic polynomial and let z_0 be a parabolic $2k$ -periodic point of $f_{c_0}(z)$. Since $f_{c_0}^{\circ 2k}$ is holomorphic, it makes sense to say that z_0 is the merger of a periodic point of period $2k$ and of points of period $2kq$. We will estimate the multiplier of the $2kq$ -periodic orbit after perturbation of c_0 to c .

Let $\rho_0 := e^{2\pi ip/q}$. By assumption, we have for z near z_0

$$f_{c_0}^{\circ 2k}(z) = z_0 + \rho_0(z - z_0) + O((z - z_0)^2) .$$

Hence it follows from the Flower Theorem [Mil06, Section 10]:

$$f_{c_0}^{\circ 2kq}(z) = z + b(z - z_0)^{q+1} + O((z - z_0)^{q+2}) .$$

The parabolic $2k$ -cycle of f_{c_0} is divided into two k -cycles of $f_{c_0}^{\circ 2}$ and each k -cycle contains one of the two critical orbits of $f_{c_0}^{\circ 2}$ in its immediate basin. Since this exhausts all the critical orbits of $f_{c_0}^{\circ 2}$, there are exactly q petals around z_0 and $b \neq 0$.

Put $z_j = f_{c_0}^{\circ j}(z_0)$ for $0 \leq j \leq 2k - 1$ and choose a fixed antiholomorphic polynomial g satisfying $g(z_j) = 0$ for all j , $g'(z_j) = 0$ for $j \neq 1$, $g'(z_1) = -f'_{c_0}(z_1)$ and $g(z) = O(\bar{z}^d)$ as $z \rightarrow 0$. Let $F_\varepsilon := f_{c_0} + \varepsilon g$ for $\varepsilon \in \mathbb{C}$. Then (z_j) is still a $2k$ -periodic cycle of F_ε . Let ρ_ε be its multiplier. Then,

$$\begin{aligned} F_\varepsilon^{\circ 2k}(z) &= z_0 + \rho_\varepsilon(z - z_0) + O((z - z_0)^2) \quad \text{and} \\ F_\varepsilon^{\circ 2kq}(z) &= z_0 + \rho_\varepsilon^q(z - z_0) + b(z - z_0)^{q+1} \\ &\quad + O(\varepsilon(z - z_0)^2) + O((z - z_0)^{q+2}) . \end{aligned}$$

We have

$$\begin{aligned} \rho_\varepsilon &= \frac{\partial}{\partial z} F_\varepsilon^{\circ 2k}(z_0) = \prod_{j=0}^{k-1} F'_\varepsilon(z_{2j+1}) \overline{F'_\varepsilon(z_{2j})} \\ &= \left(1 + \varepsilon \frac{g'(z_1)}{f'_{c_0}(z_1)}\right) \prod_{j=0}^{k-1} f'_{c_0}(z_{2j+1}) \overline{f'_{c_0}(z_{2j})} = (1 - \varepsilon)\rho_0 \end{aligned}$$

and $\rho_\varepsilon^q = 1 - q\varepsilon + O(\varepsilon^2)$; here F'_ε denotes the antiholomorphic derivative $\frac{\partial}{\partial \bar{z}} F_\varepsilon$, and similarly for f'_{c_0} .

Any $2kq$ -periodic point z of F_ε bifurcating from z_0 satisfies

$$\begin{aligned} 0 &= F_\varepsilon^{\circ 2kq}(z) - z \\ &= z_0 - z + \rho_\varepsilon^q(z - z_0) + b(z - z_0)^{q+1} + O(\varepsilon(z - z_0)^2) + O((z - z_0)^{q+2}) \\ &= (z - z_0) \left(-1 + 1 - q\varepsilon + O(\varepsilon^2) + b(z - z_0)^q \right. \\ &\quad \left. + O(\varepsilon(z - z_0)) + O((z - z_0)^{q+1}) \right) . \end{aligned} \tag{3.1}$$

There are clearly q such bifurcating points near z_0 . To stress dependence on ε , we write z_ε for z and get $b(z_\varepsilon - z_0)^q = q\varepsilon(1 + o(1))$ and hence

$$z_\varepsilon = z_0 + (q\varepsilon/b)^{1/q} (1 + o(1)) \quad \text{as } \varepsilon \rightarrow 0 .$$

The multiplier is

$$\begin{aligned} \tilde{\rho}_\varepsilon &= \left(\frac{\partial}{\partial z} F_\varepsilon^{\circ 2kq} \right) (z_\varepsilon) = \rho_\varepsilon^q + b(q+1)(z_\varepsilon - z_0)^q + o(\varepsilon) \\ &= 1 - q\varepsilon + (q+1)q\varepsilon + o(\varepsilon) = 1 + q^2\varepsilon + o(\varepsilon) . \end{aligned}$$

For sufficiently small $\varepsilon < 0$ we have $|\tilde{\rho}_\varepsilon| < 1$, so z_ε is an attracting periodic point of F_ε .

The rest of the proof is as usual: restrict f_{c_0} to an anti-polynomial-like map of degree d and make ε small enough so that after perturbation we still get an anti-polynomial-like map of degree d ; straightening yields a unicritical antiholomorphic polynomial $\bar{z}^d + c(\varepsilon)$, and $c(\varepsilon)$ depends continuously on ε with $c(0) = c_0$ (continuity is not in general true for straightening, but our construction assures that the quasi-conformal gluing map has dilatation tending to zero as $\varepsilon \rightarrow 0$).

Note that we have q choices of the q -th root in z_ε , but the multiplier is independent of this choice (at least to leading order). All q choices of z_ε will thus be attracting simultaneously. Recall that z_0 had period $2k$. Since our polynomials $f_{c(\varepsilon)}$ can only have a single non-repelling cycle, all these attracting periodic points must be on the same cycle, which therefore has period at least $2qk$; the period cannot exceed $2qk$ because of (3.1). This completes the proof. $\square$

For odd k , bifurcations have an entirely different form; see Section 3.4.

Lemma 3.1.2 (Landing of Dynamical Rays). *For every map f_c and every periodic angle t of some period n , the dynamical ray $R_c(t)$ lands at a repelling periodic point of period dividing n , except in the following circumstances:*

- $c \in \mathcal{M}_d^*$ and $R_c(t)$ lands at a parabolic cycle;
- $c \notin \mathcal{M}_d^*$ is on the parameter ray at some angle $(-d)^k t$, for $k \in \mathbb{N}$.

Conversely, every repelling or parabolic periodic point is the landing point of at least one periodic dynamical ray for every $c \in \mathcal{M}_d^$.*

Proof. This is well known in the holomorphic case, and can be proved analogously in the antiholomorphic setting. For the case $c \in \mathcal{M}_d^*$ (i.e. when the Julia set is connected), see [Mil06, Theorem 18.10]; if $c \notin \mathcal{M}_d^*$ (i.e. when the Julia set is disconnected), see [DH85a, Exposé n° VIII, §II, Proposition 1], [GM93, Appendix A]. For the converse, see [Mil06, Theorem 18.11]. $\square$

We should note that for every periodic point z_0 of period p of f_{c_0} with multiplier $\lambda(c_0, z_0) \neq 1$, the orbit of z_0 remains locally stable under perturbation of the parameter. With some restrictions, this is even true for the dynamical rays of f_{c_0} that land at z_0 .

Lemma 3.1.3 (Stability of Landing Rays). *Let c_0 be a parameter such that $f_{c_0} = \bar{z}^d + c$ has a repelling periodic point z_0 of period p so that the dynamical rays $R_{c_0}(t)$ with $t \in A := \{t_1, \dots, t_v\}$ ($v \geq 2$) land at z_0 .*

1) Then there exists a neighborhood U of c_0 , and a unique real-analytic function $z: U \rightarrow \mathbb{C}$ with $z(c_0) = z_0$ so that for every $c \in U$ the point $z(c)$ is

a repelling periodic point of period p for f_c , and all the dynamical rays $R_c(t)$ with $t \in A$ land at $z(c)$.

2) Let $\{n_1, n_2, \dots, n_v\}$ be the periods of the rays in A . Define F to be the set of all parameters c such that f_c has a parabolic cycle, and some dynamical ray of period n_i ($i \in \{1, 2, \dots, v\}$) lands on the parabolic cycle of f_c . Let U be an open, connected neighborhood of c_0 which is disjoint from F and from all parameter rays at angles in $\tilde{A} := \bigcup_{j \geq 0} (-d)^j A$. Then for every parameter $c \in U$, the dynamical rays $R_c(t)$ with $t \in A$ land at a common point.

Proof. We work out the case when p is even. When p is odd, we simply need to work with the holomorphic first return map $f_{c_0}^{\circ 2p}$ of z_0 .

1) Since z_0 is a repelling periodic point of f_{c_0} , the multiplier $\lambda(c_0, z_0) := (f_{c_0}^{\circ p})'(z_0)$ is greater than 1 in absolute value. By the Implicit Function Theorem, we can solve the equation $F(c, z) := f_c^{\circ p}(z) - z = 0$ for the periodic point $z(c)$ (observe that $\frac{\partial F}{\partial z}|_{(c_0, z_0)} = (f_{c_0}^{\circ p})'(z_0) - 1 = \lambda(c_0, z_0) - 1 \neq 0$) as a real-analytic function of c in a neighborhood U of c_0 . The multiplier $(f_c^{\circ p})'(z(c))$ of f_c at $z(c)$ is a real-analytic function as well, and hence is greater than 1 in absolute value for c close to c_0 . This proves that the point z_0 can be continued real-analytically as a repelling periodic point $z(c)$ of f_c in a neighborhood U of c_0 . Arguing as in [GM93, Lemma B.1], [Sch00, Lemma 2.2], one can show that the dynamical rays $R_c(t)$ with $t \in A$ land at the real-analytically continued periodic point $z(c)$ (possibly after shrinking U).

2) Since U does not intersect any parameter ray at angles in $\tilde{A}$, Lemma 3.1.2 tells that for all $c \in U$, the dynamical rays $R_c(t)$ with $t \in A$ indeed land. Let $U' := \{c \in U : \text{the dynamical rays } R_c(t) \text{ with } t \in A \text{ land at a common point}\}$. For any $c \in U'$, the common landing point of the dynamical rays $R_c(t)$ with $t \in A$ must be repelling as $U \cap F = \emptyset$. By part (1), we conclude that U' is open in U .

Let $\tilde{c}$ be a limit point of U' in the subspace topology of U . The dynamical rays $R_{\tilde{c}}(t)$ with $t \in A$ do land in the dynamical plane of $f_{\tilde{c}}$; we claim that they all land at a common point. Otherwise, at least two rays $R_{\tilde{c}}(t_1)$ and $R_{\tilde{c}}(t_2)$ (say) would land at two different repelling periodic points z_1 and z_2 (respectively) in the dynamical plane of $f_{\tilde{c}}$. Choose neighborhoods V_1 and V_2 of z_1 and z_2 respectively such that $V_1 \cap V_2 = \emptyset$. Applying part (1) to this situation, we see that for c close to $\tilde{c}$, the landing points of $R_c(t_1)$ and $R_c(t_2)$ would lie in V_1 and V_2 respectively; i.e. the dynamical rays $R_c(t_1)$ and $R_c(t_2)$ would land at different points. This contradicts the fact that $\tilde{c}$ is a limit point of U' . Hence, $\tilde{c} \in U'$, and U' is closed in U . Since U is connected, $U' = U$. This completes the proof of the lemma. $\square$

Corollary 3.1.4 (Stability of Rays at Bifurcations). *Let c_0 has an indifferent cycle of even period, and H be a hyperbolic component with $c_0 \in \partial H$. Let the dynamical rays $R_{c_0}(\vartheta)$ with $\vartheta \in \{\vartheta_1, \dots, \vartheta_r\}$ land at a common repelling*

periodic point of f_{c_0} . Then for all $c \in H$, the dynamical rays $R_c(\vartheta)$ with $\vartheta \in \{\vartheta_1, \dots, \vartheta_r\}$ land at a common repelling periodic point of f_c .

If further, the non-repelling cycle of c_0 is parabolic, and if the dynamical rays $R_{c_0}(t)$ with $t \in A := \{t_1, \dots, t_v\}$ land at a parabolic periodic point z_0 of f_{c_0} , then for all $c \in H$, the dynamical rays $R_c(t)$ with $t \in A$ land at a common repelling cycle of f_c . The period of this repelling cycle may be equal or greater than the period of the parabolic cycle. If the periods are equal, then for all $c \in H$, the dynamical rays $R_c(t)$ with $t \in A$ land at a common repelling periodic point of f_c .

Proof. Any arbitrarily small perturbation of c_0 intersect H , and hence by part (1) of Lemma 3.1.3, the dynamical rays $R_c(\vartheta)$ with $\vartheta \in \{\vartheta_1, \dots, \vartheta_r\}$ land at a common repelling periodic point of f_c for any $c \in B(c_0, \varepsilon) \cap H$ (for $\varepsilon > 0$ small enough). But the hyperbolic component H does not intersect any parabolic parameter or any parameter ray. Therefore, by part (2) of Lemma 3.1.3, the required property continues to hold throughout H .

Since the period of the indifferent cycle is even, the second fact can be proved as in the holomorphic setting. We refer the readers to [Mil00b, Theorem 4.1, Lemma 4.4, Lemma 6.1] for the details. $\square$

Remark 3.1.5. *If the period of the parabolic cycle is odd, more care is needed. We will show later that the rays landing at an odd periodic parabolic point can be shared by two distinct repelling cycles after perturbation.*

Before we state our next lemma, we need a brief digression to algebra. It is well-known that the *resultant* of two univariate polynomials P and Q is a polynomial in the coefficients of P and Q which vanishes if and only if P and Q have a common root; i.e. if and only if $\deg(\gcd(P, Q)) \geq 1$. It is sometimes desirable to generalize the notion of *resultants* to be able to predict the exact degree of the gcd of P and Q in terms of their coefficients. This can be done by the so-called *subresultants*, which are polynomials in the coefficients of P and Q , and whose order of vanishing tells us the degree of the gcd of the two polynomials P and Q . We refer the readers to [BPR06, §4.2.2] for the precise definition of subresultants.

The main result on *subresultants* that will be used in the proof of the Lemma 3.1.7 is the following:

Lemma 3.1.6. *For two univariate polynomials P and Q over a domain (of degree p and q respectively, such that $p > q$), $\deg(\gcd(P, Q)) \geq j$ for some $0 \leq j \leq q$ if and only if $\text{sRes}_0(P, Q) = \dots = \text{sRes}_{j-1}(P, Q) = 0$, where each $\text{sRes}_i(P, Q)$ is a polynomial expression in the coefficients of P and Q .*

Proof. See [BPR06, Proposition 4.25]. $\square$

We denote the unit disc in the complex plane by $\mathbb{D}$ and the set of all roots of unity by U .

Lemma 3.1.7 (Multipliers Isolated). *Let $k \in \mathbb{Z}^+$ and $\mu \in \mathbb{D} \cup U$. Then the set of parameters $c \in \mathbb{C}$ for which $\bar{z}^d + c$ has a periodic orbit with period $2k$ and multiplier μ is finite.*

Proof. We embed the family $\{f_c\}$ in a two-parameter family of polynomials $\{P_{a,b}(z) = (z^d + a)^d + b; (a, b) \in \mathbb{C}^2\}$ which depends analytically on the parameters a and b . The critical points of these polynomials are 0 and the d roots of the equation $z^d + a = 0$ such that each critical point has multiplicity $d - 1$; however, these maps have only two critical values $a^d + b$ and b . Since $f_c^{\circ 2} = P_{\bar{c},c}$, our family can be regarded as a real 2-dimensional slice (namely, $a = \bar{b}$) of the family $\{P_{a,b}\}_{a,b \in \mathbb{C}}$. A $2k$ -cycle (z_j) (where $z_j = f_c^{\circ j}(z_0), 0 \leq j \leq 2k-1$) of f_c gives two distinct k -cycles (z_{2i}) and (z_{2i+1}) (for $i = 0, \dots, k-1$) of $f_c^{\circ 2}$; the multipliers of these two cycles are easily seen to be complex conjugates of each other; denote them by μ and $\bar{\mu}$.

We define the following sets:

- $\Lambda := \{c \in \mathbb{C} : f_c \text{ has a periodic orbit of period } 2k \text{ with multiplier } \mu\},$
- $\Lambda(k, \mu) := \{(a, b) \in \mathbb{C}^2 : P_{a,b} \text{ has two distinct cycles of period dividing } k \text{ with multipliers } \mu \text{ and } \bar{\mu}\},$
- $\Lambda'(k, \mu) = \{(a, b) \in \mathbb{C}^2 : P_{a,b} \text{ has two distinct cycles of period } k \text{ with multipliers } \mu \text{ and } \bar{\mu}\}.$

Clearly, $\Lambda \subset \{c \in \mathbb{C} : (\bar{c}, c) \in \Lambda'(k, \mu)\}$, $|\{c \in \mathbb{C} : (\bar{c}, c) \in \Lambda'(k, \mu)\}| = |\Lambda'(k, \mu) \cap \{(a, b) \in \mathbb{C}^2 : a = \bar{b}\}|$ and $\Lambda'(k, \mu) \subset \Lambda(k, \mu)$.

Case 1. First assume that $\mu \in (\mathbb{D} \cup U) \setminus \mathbb{R}$. We will show that the set $\Lambda(k, \mu)$ is finite. Let us consider the complex numbers a, b, z_0, z_1 satisfying the four algebraic equations

$$P_{a,b}^{\circ k}(z_0) - z_0 = 0, \quad (P_{a,b}^{\circ k})'(z_0) = \mu, \quad (3.2)$$

$$P_{a,b}^{\circ k}(z_1) - z_1 = 0, \quad (P_{a,b}^{\circ k})'(z_1) = \bar{\mu}. \quad (3.3)$$

Let $R_\mu(a, b)$ (respectively $R_{\bar{\mu}}(a, b)$) be the resultant of the two polynomials $P_{a,b}^{\circ k}(z) - z$ and $(P_{a,b}^{\circ k})'(z) - \mu$ (respectively $P_{a,b}^{\circ k}(z) - z$ and $(P_{a,b}^{\circ k})'(z) - \bar{\mu}$). Then a solution z_0 of (3.2) (respectively z_1 of (3.3)) exists if and only if $R_\mu(a, b) = 0$ (respectively $R_{\bar{\mu}}(a, b) = 0$). It follows that $\Lambda(k, \mu) = \{R_\mu(a, b) = R_{\bar{\mu}}(a, b) = 0\}$. Since $\mu \notin \mathbb{R}$, the two algebraic varieties $R_\mu(a, b) = 0$ and $R_{\bar{\mu}}(a, b) = 0$ are distinct. By Bézout's theorem, the intersection of two algebraic varieties $R_\mu(a, b) = 0$ and $R_{\bar{\mu}}(a, b) = 0$ is either a finite set or a common irreducible component with unbounded projection over each variable. If they have a common irreducible component, say S , then $S \subset \Lambda(k, \mu) = \{R_\mu(a, b) = R_{\bar{\mu}}(a, b) = 0\}$. This would force $\Lambda(k, \mu)$ to be unbounded. But since $\mu \in \mathbb{D} \cup U$, for any $(a, b) \in \Lambda(k, \mu)$, $P_{a,b}$ has two distinct attracting/parabolic cycles with multipliers μ and $\bar{\mu}$. As a result, these periodic orbits would attract at least one

infinite critical orbit each. This forces all the critical orbits of $P_{a,b}$ to stay bounded. This implies that $\Lambda(k, \mu)$ is contained in the connectedness locus of the family of monic centered polynomials of degree d^2 , which is compact by [BH88]: a contradiction!

Case 2. Now we consider $\mu \in [-1, 1)$. We no longer have two distinct algebraic varieties given by (3.2) and (3.3) and the arguments of the previous case do not work. To circumvent this problem, we use the theory of *subresultants*.

Note that for any $(a, b) \in \Lambda'(k, \mu)$, the two polynomial equations $P_{a,b}^{\circ k}(z) - z = 0$ and $(P_{a,b}^{\circ k})'(z) - \mu = 0$ (viewed as equations in $\mathbb{C}[a, b][z]$) have exactly $2k$ distinct common solutions, all of which are simple roots of $P_{a,b}^{\circ k}(z) - z = 0$. Thus their gcd is a polynomial in $\mathbb{C}[a, b][z]$ of degree $2k$. Note that in this case, the polynomial $P_{a,b}$ can have at most two cycles of multiplier μ . It follows from Lemma 3.1.6 that:

$$\begin{aligned} \Lambda'(k, \mu) &= \{(a, b) \in \mathbb{C}^2 : \text{sRes}_0 \left(P_{a,b}^{\circ k}(z) - z, (P_{a,b}^{\circ k})'(z) - \mu \right) = \dots \\ &= \text{sRes}_{2k-1} \left(P_{a,b}^{\circ k}(z) - z, (P_{a,b}^{\circ k})'(z) - \mu \right) = 0\}. \end{aligned} \quad (3.4)$$

It again follows from Bézout's theorem that the intersection of these algebraic varieties is a finite set: if there was a common irreducible component, then this component would be contained in $\Lambda'(k, \mu)$ forcing it to be unbounded. But the two critical orbits of $P_{a,b}$ must be attracted by the two (attracting or parabolic) cycles of period k , implying that $\Lambda'(k, \mu)$ is contained in the connectedness locus of the family of polynomials of degree d^2 , which is compact by [BH88]: a contradiction!

Case 3. Finally we consider $\mu = 1$. For any $(a, b) \in \Lambda'(k, 1)$, $P_{a,b}$ has two distinct parabolic cycles of period k and multiplier 1. The first return map $P_{a,b}^{\circ k}$ of each of these cycles fixes the petals. Since $P_{a,b}$ has only two infinite critical orbits and each cycle of petals absorbs at least one infinite critical orbit, there is exactly one petal associated with each of these parabolic periodic points. In other words, each of these parabolic periodic points is a double root of $P_{a,b}^{\circ k}(z) - z$. Hence, each of them contributes a linear factor to the gcd of $P_{a,b}^{\circ k}(z) - z$ and $(P_{a,b}^{\circ k})'(z) - 1$. Since $P_{a,b}$ can not have any more parabolic cycles, the g.c.d of $P_{a,b}^{\circ k}(z) - z$ and $(P_{a,b}^{\circ k})'(z) - 1$ is a polynomial of degree $2k$. Thus,

$$\begin{aligned} \Lambda'(k, 1) &\subset \{(a, b) \in \mathbb{C}^2 : \text{sRes}_0 \left(P_{a,b}^{\circ k}(z) - z, (P_{a,b}^{\circ k})'(z) - 1 \right) = \dots \\ &= \text{sRes}_{2k-1} \left(P_{a,b}^{\circ k}(z) - z, (P_{a,b}^{\circ k})'(z) - 1 \right) = 0\}. \end{aligned} \quad (3.5)$$

By Bézout's theorem, this intersection of algebraic curves is either a finite set of points or contains an unbounded irreducible component. In addition to $\Lambda'(k, 1)$, the right hand side of (3.5) can possibly contain some additional points of the form:

S_1 : Parameters (a, b) such that $P_{a,b}$ has a single parabolic cycle of period k' (for some k' dividing k) and multiplier a k/k' -th root of unity such that there are two cycles of petals for the parabolic cycle.

S_2 : Parameters (a, b) such that $P_{a,b}$ has two distinct parabolic cycles of period k' and k'' (for some k' and k'' less than k and dividing k) with multipliers some k/k' -th and k/k'' -th roots of unity respectively, such that each parabolic cycle has a single cycle of petals associated with it.

Observe that for any $(a, b) \in \Lambda'(k, 1) \cup S_1 \cup S_2$, the polynomial $P_{a,b}$ has both its critical orbits bounded. In other words, the right hand side of (3.5) is contained in the connectedness locus of monic centered polynomials of degree d^2 . Hence the intersection of the algebraic curves in (3.5) can not contain an unbounded irreducible component and hence is finite. This finishes the proof of the lemma. $\square$

Remark 3.1.8. *The restriction to even periods k is essential: for odd k , the period k orbit of the antiholomorphic polynomial f_c does not split into two separate period k orbits of $P_{\bar{c},c}$, so the given proof does not work. In fact, the set of parameters for which there is an orbit of odd period k with multiplier $\mu = +1$ contains finitely many arcs: this set is equal to the union of the boundaries of the hyperbolic components of period k ; see Theorem 3.2.1 below.*

Lemma 3.1.9 (Indifferent Dynamics of Odd Period). *The boundary of a hyperbolic component of odd period k consists entirely of parameters having a parabolic cycle of period k . In local conformal coordinates, the $2k$ -th iterate of such a map has the form $z \mapsto z + bz^{q+1} + \dots$ with $q \in \{1, 2\}$.*

Proof. Let H be a hyperbolic component of odd period, and $c \in \partial H$. Then, f_c has an indifferent cycle of period k' dividing k (note that k' is odd). The second iterate $f_c^{\circ 2k'}$ of the first return map $f_c^{\circ k'}$ of a periodic point of odd period k' has a non-negative real multiplier. Since the cycle is indifferent, the absolute value of its multiplier is $+1$, and hence, the multiplier is always $+1$. This shows that every $c \in \partial H$ has a parabolic cycle of period k' (dividing k) and of multiplier 1.

The holomorphic first return map $f_c^{\circ 2k'}$ of such a parabolic periodic point has the local form: $z \mapsto z + bz^{q+1} + O(z^{q+2})$, where q is the number of attracting petals attached to the parabolic periodic point [Mil06, §10]. Since the multiplier of the parabolic point is $+1$, $f_c^{\circ 2k'}$ fixes each petal (compare [NS03, Corollary 4.2]), and hence there are precisely q cycles of petals under $f_c^{\circ 2}$. Each of these cycles absorbs an infinite critical orbit of $f_c^{\circ 2}$. But $f_c^{\circ 2}$ has only two infinite critical orbits, and hence $q \in \{1, 2\}$.

If the period k' of the parabolic cycle is strictly smaller than k (i.e. if H is a ‘satellite’ component), then c lies on the boundary of a hyperbolic component H' of period k' (by Theorem 3.1.1). So, exactly k/k' points of the k -periodic attracting cycle of H would coalesce (along with a repelling periodic point) at the parabolic parameter c , giving rise to k/k' petals at each parabolic periodic

point of f_c . In other words, the multiplicity of any parabolic point of f_c would be $(k/k' + 1)$; i.e. $q = k/k'$. As k/k' is necessarily odd, and $q \in \{1, 2\}$, we have $k/k' = 1$, i.e. $k = k'$. This completes the proof of the lemma. $\square$

An orbit is parabolic if it is a merger of at least two periodic orbits, which generally is a co-dimension one condition. It turns out that for odd periods, this gives only a real co-dimension, so there are entire arcs of parabolic parameters (Theorem 3.2.1). Cusps are a merger of three periodic orbits, and this has real co-dimension two: there are only finitely many cusps.

Lemma 3.1.10 (Finitely Many Cusp Points). *The number of cusp points of any given (odd) period is finite.*

Proof. We use a similar idea as in the proof of Lemma 3.1.7. Let k be odd and consider again the family $P_{a,b}(z) = (z^d + a)^d + b$ for complex parameters a and b . Observe that if c is a cusp parameter (with the period of the parabolic cycle being k), then the second iterate $f_c^{\circ 2}$ is a polynomial with a parabolic periodic point of period k and multiplicity 2 (in other words, $P_{\bar{c},c}$ has a double parabolic periodic point of period k). Thus, it suffices to prove the finiteness assertion in the family $P_{a,b}$. Suppose a , b , and z be such that z is a parabolic periodic point of period k and multiplicity 2 for $P_{a,b}$; then we have the following three equations:

$$\begin{aligned} P_{a,b}^{\circ k}(z) - z &= 0 && \text{to have a fixed point} \\ (P_{a,b}^{\circ k})'(z) &= 1 && \text{to make it parabolic} \\ (P_{a,b}^{\circ k})''(z) &= 0 && \text{to make it a cusp.} \end{aligned}$$

The set of simultaneous solutions to these three equations is, again using Bézout's Theorem, either finite or an algebraic variety with unbounded projection over every variable.

The polynomial $P_{a,b}$ has two infinite critical orbits. If the map $P_{a,b}$ satisfies the previous three conditions, then $P_{a,b}$ has two distinct cycles of petals (recall that the multiplier of the parabolic cycle is $+1$), and each cycle of petals would attract an infinite critical orbit. Thus, both critical orbits would converge to the parabolic cycle; so they both would be bounded. Therefore, all the cusps are contained in the connectedness locus of monic centered polynomials of degree d^2 , which is compact. Therefore, the set of solutions to the above three equations is finite. This proves the lemma. $\square$

Corollary 3.1.11 (Restricted Bifurcations from Odd Periods). *If f_c has an indifferent cycle of odd period k , and c is on the boundary of a hyperbolic component (contained in $\mathcal{M}_d^*$) of period n , then $n \in \{k, 2k\}$.*

Proof. If c is on the boundary of a hyperbolic component W of period n , then n must be a multiple of k ; setting $q := n/k$, then q points each of the n -periodic attracting cycle in W coalesce at c . The holomorphic second iterate

$f_c^{\circ 2k}$ of the first return map $f_c^{\circ k}$ then has the form $z \mapsto z + z^{q+1} + \dots$ in local coordinates, and $q \in \{1, 2\}$ by Lemma 3.1.9. $\square$

Lemma 3.1.12 (Finitely Many Hyperbolic Components). *For every (even or odd) period k , the number of hyperbolic components of $\mathcal{M}_d^*$ with period k is finite.*

Proof. The structure theorem on hyperbolic components in [NS03] says that each even or odd period component has a unique center at which the critical orbit is periodic. Then, for the even period case, the claim follows from Lemma 3.1.7. For the odd period case, we take the second iterate and it suffices to show that there are only finitely many parameters (a, b) for which both critical orbits of $(z^d + a)^d + b$ are periodic with period k . The corresponding Julia sets are connected, so the claim follows once again from Bézout's theorem. $\square$

Remark 3.1.13. *a) The finiteness can also be shown, possibly more dynamically, using Hubbard trees.*

b) In Section 3.6, we will give a recursive formula for the number of hyperbolic components of a given period.

Lemma 3.1.14 (Orbit Separation Lemma). *For every anti-polynomial $f(z) = \bar{z}^d + c$ with parabolic dynamics, there are two periodic or pre-periodic dynamical rays which land at a common point and which together separate the characteristic parabolic point from the rest of the parabolic orbit.*

Proof. The proof is similar to the holomorphic case, see [Sch00, Lemma 3.7]. $\square$

3.2 Parabolic Arcs

We mentioned in the introduction that the critical Ecalle height of a non-cusp odd period parabolic parameter is a conformal conjugacy invariant, and contains crucial information about the global geometry of the immediate parabolic basin. The purpose of this section is to show that one can change the Ecalle height by a quasi-conformal deformation to yield real-analytic curves of quasi-conformally conjugate non-cusp parabolic parameters. The proof is reminiscent of Branner and Hubbard's wringing construction, where one changes the conformal structure in a fundamental domain (of the dynamics) to modify the conformal position of the critical value [BH88, §II].

Theorem 3.2.1 (Parabolic Arcs). *Let c_0 be a parameter such that f_{c_0} has a parabolic cycle of odd period, and suppose that c_0 is not a cusp. Then c_0 is on a parabolic arc in the following sense: there exists a simple real-analytic arc of parabolic parameters $c(t)$ (for $t \in \mathbb{R}$) with quasi-conformally equivalent but conformally distinct dynamics of which c_0 is an interior point.*

This result has first been shown by Nakane [Nak93b] using different methods.

Proof. We will use quasi-conformal (qc-) deformations. The critical orbit is contained in the parabolic basin. We parametrize the horizontal coordinate within the Ecalle cylinder by $\mathbb{R}/\mathbb{Z}$. Choose the horizontal Fatou coordinate (the last degree of freedom of the Fatou coordinate) so that the critical value has real part $1/4$ within the Ecalle cylinder. Let the critical Ecalle height of f_{c_0} be h .

It is easy to change the complex structure within the Ecalle cylinder so that the critical Ecalle height becomes any assigned real value, for example via the quasi-conformal homeomorphism

$$\ell_t : (x, y) \mapsto \begin{cases} (x, y + 2tx) & \text{if } 0 \leq x \leq 1/4 \\ (x, y + t(1 - 2x)) & \text{if } 1/4 \leq x \leq 1/2 \\ (x, y - t(2x - 1)) & \text{if } 1/2 \leq x \leq 3/4 \\ (x, y - 2t(1 - x)) & \text{if } 3/4 \leq x \leq 1 \end{cases}$$

This homeomorphism ℓ_t commutes with the map $I : z \mapsto \bar{z} + 1/2$, hence the corresponding Beltrami form is invariant under the map I . Note that $\ell_t(1/4, h) = (1/4, h + t/2)$. Translating the map ℓ_t by positive integers, we obtain a qc-map ℓ_t commuting with I in a right half plane.

By the coordinate change $z \mapsto \Phi_0(z)$ (where Φ_0 is the attracting Fatou coordinate at the characteristic parabolic point of f_{c_0}), we can transport this Beltrami form from the right half plane into all the attracting petals, and it is forward invariant under f_{c_0} . It is easy to make it backward invariant by pulling it back along the dynamics. Extending it by the zero Beltrami form outside of the entire parabolic basin, we obtain an invariant Beltrami form, and the Measurable Riemann Mapping Theorem supplies a qc-map φ_t integrating this Beltrami form and conjugating the original map to a new antiholomorphic polynomial $f_{c(t)}$ within the same family. Its attracting Fatou coordinate at the characteristic parabolic point is given by $\Phi_t = \ell_t \circ \Phi_0 \circ \varphi_t^{-1}$. Thus the critical Ecalle height of $f_{c(t)}$ is $h + t/2$.

Note that the Beltrami form depends real analytically on t , so the parameter $c(t)$ depends real analytically on t . We obtain a real analytic map from $\mathbb{R}$ into the multicorn $\mathcal{M}_d^*$. Since the critical points of all $f_{c(t)}$ have different Ecalle heights, which is a conformal invariant, this map is injective. This proves the existence of an arc in the parameter plane with the required properties, and injectivity implies that the arc is simple. $\square$

Remark 3.2.2. 1. *Since all the antiholomorphic polynomials f_c on a parabolic arc $\mathcal{C}$ have quasi-conformally conjugate dynamics, there exists a fixed odd integer k such that each f_c on $\mathcal{C}$ has a k -periodic parabolic cycle. Such an arc will be referred to as a parabolic arc of period k .*

2. By our construction, the critical Ecalle height of $f_{c(t)}$ is $h + t/2$. Therefore, the critical Ecalle height of $f_{c(t)}$ tends to $\pm\infty$ towards the ends of the arc; i.e. as $t \rightarrow \pm\infty$.

3. Numerical experiments suggest that the arc is a smooth curve in the parameter plane when parametrized by arc-length. This would follow if we could prove that the map $c(t)$ had a nowhere vanishing derivative. One can, by passing to the biquadratic family, show that there are at most (possibly) finitely many singular points of this parametrization (see Chapter 4). The question whether such finitely many exceptional points indeed exist, is related to the smoothness of certain algebraic curves and requires further investigation.

Lemma 3.2.3 (Endpoints of Parabolic Arcs). *For every parabolic arc $c(t)$, the limits $c^\pm := \lim_{t \rightarrow \pm\infty} c(t)$ exist and are parabolic cusps of the same period.*

Proof. The parabolic arc $t \mapsto c(t)$ is a continuous map $\mathbb{R} \rightarrow \mathcal{M}_d^*$, so the limit sets $L^\pm$ of all accumulation points as $t \rightarrow \pm\infty$ are connected and non-empty.

Let k be the odd period of the parabolic orbit for all $c(t)$. By continuity, every $c \in L^\pm$ is parabolic with period at most k . By Lemma 3.1.9, the period of the parabolic orbit of c is exactly k .

Note that the critical Ecalle height depends continuously on the parameter. In fact, the construction of Fatou coordinates in lemma 1.1.14 depends locally uniformly on the parameter around any non-cusp c . Therefore, if $c \in L^\pm$ is not cusp, the critical Ecalle heights for $c(t_n)$ tending to c , are bounded, which is a contradiction. Thus $L^\pm$ consists of cusps. By finiteness of the number of cusps, the claim follows. $\square$

Lemma 3.2.4 (Parabolic Arcs Disjoint). *No two parabolic arcs have any point in common, other than their endpoints.*

Proof. Let $c_1(t)$ and $c_2(t)$ be two parabolic arcs, parametrized so that $c_i(t)$ has critical Ecalle height t for every $t \in \mathbb{R}$. If these arcs have an interior point in common, then $c_1(t_0) = c_2(t_0)$ for some $t_0 \in \mathbb{R}$, and all $c_1(t)$ and all $c_2(t')$ are quasi-conformally conjugate to $c_i(t_0)$ and hence to each other. For every $t \in \mathbb{R}$, the quasi-conformal conjugation between $c_1(t)$ and $c_2(t)$ is conformal on the Ecalle cylinder (by identical critical Ecalle height) and hence on every bounded Fatou component, and it is also conformal on the basin of infinity. Since the Julia set has measure zero for every polynomial for which all critical orbits are in parabolic basins, $c_1(t)$ and $c_2(t)$ are conformally conjugate, and $c_1(t) = c_2(t)$ for all t because $c_1(t_0) = c_2(t_0)$. $\square$

However, the closures of two distinct parabolic arcs may intersect.

Corollary 3.2.5 (Neighborhoods of Arcs). *For every parabolic arc $\mathcal{C}$ of odd period k , there is a unique hyperbolic component H of period k such that $\mathcal{C} \subset \partial H$. The arc $\mathcal{C}$ does not intersect the boundary of any other hyperbolic component of period k .*

Proof. Let us assume that such an H does not exist, and we will obtain a contradiction. By Theorem 3.1.1, each point of $\mathcal{C}$ lies on the boundary of a hyperbolic component of period k of $\mathcal{M}_d^*$. Since there are only finitely many hyperbolic components of period k , our assumption implies that there are distinct hyperbolic components $H_1, H_2, \dots, H_r$ of period k such that $\mathcal{C} \subset \partial H_1 \cup \partial H_2 \cup \dots \cup \partial H_r$, but $\mathcal{C} \not\subset \partial H_i$ for $i = 1, 2, \dots, r$. It suffices to consider the case when $r = 2$.

Let $c : \mathbb{R} \rightarrow \mathcal{C}$ be the critical Ecalle height parametrization of $\mathcal{C}$ (given in Theorem 3.2.1). Then there exists $t_0 \in \mathbb{R}$ such that $c((-\infty, t_0]) \subset \partial H_1$ and $c([t_0, \infty)) \subset \partial H_2$. Since H_1 is homeomorphic to $\mathbb{D}$ [NS03, Lemma 5.4, Theorem 5.9], and every parabolic arc limits at cusp points on both ends (by Lemma 3.2.3), ∂H_1 is a closed curve consisting of (possibly parts of) parabolic arcs and cusp points. Since each parabolic arc is a simple arc (i.e. there is no self-intersection), this implies that there exists a parabolic arc $\mathcal{C}'$ (intersecting ∂H_1), different from $\mathcal{C}$, such that the parameter $c(t_0) \in \mathcal{C} \cap \mathcal{C}'$. This contradicts Lemma 3.2.4, and proves the existence of a hyperbolic component H of period k with $\mathcal{C} \subset \partial H$.

The above argument also shows that if some $c \in \mathcal{C}$ lies on ∂H and $\partial H'$, for two distinct hyperbolic components H and H' of period k , then $\mathcal{C} \subset \partial H \cap \partial H'$. But this is impossible because, by Theorem 3.1.1, every neighborhood of every point on a parabolic arc meets parameters where all orbits of period k are repelling. Therefore, $\mathcal{C}$ does not intersect the boundary of any hyperbolic component of period k , other than the boundary of H . The uniqueness follows. $\square$

Definition 3.2.6 (Root Arcs and Co-Root Arcs). *We call a parabolic arc a root arc if for every parameter c on this arc, the parabolic cycle of f_c disconnects the Julia set of f_c (if $(z_i)_{1 \leq i \leq k}$ is the parabolic cycle of f_c , then we say that $(z_i)_{1 \leq i \leq k}$ disconnects $J(f_c)$ if $J(f_c) \setminus (z_i)$ is disconnected). Otherwise, we call it a co-root arc.*

Remark 3.2.7. *Since the dynamics of all the points on the arc are quasi-conformally conjugate, this classification of arcs is well-defined.*

It follows from Corollary 3.1.11 and Corollary 3.2.5 that any hyperbolic component bifurcating from a hyperbolic component of odd period k must have period $2k$. This possibility is indeed realized at the end of every parabolic arc, and the corresponding bifurcation occurs along an arc of positive length. We recall the precise statement, a proof of which can be found in [HS14, Theorem 3.8, Corollary 3.9].

Theorem 3.2.8 (Bifurcations Along Arcs). *Every parabolic arc has, at both ends, an interval of positive length at which a bifurcation from a hyperbolic component of odd period k to a hyperbolic component of period $2k$ occurs.*

3.3 Parabolic Arcs and Orbit Portraits

In this section, we study the combinatorial properties of the parabolic arcs in details.

Lemma 3.3.1 (Number of Root and Co-roots). *1) Every dynamical co-root is the landing point of exactly one dynamical ray, and this ray has the same period as the Fatou component.*

2) Every periodic Fatou component (for a unicritical antiholomorphic polynomial f_c) of period greater than 1 corresponding to an attracting/parabolic cycle has exactly one root. If the period of the component is even; then it has exactly $d - 2$ co-roots and if the period is odd; it has exactly d co-roots. Every Fatou component of period 1 has exactly $d + 1$ co-roots and no root.

Proof. See [NS03, Lemma 3.4]. □

Lemma 3.3.2 (Disjoint Closures of Fatou Components). *Let H be a hyperbolic component of odd period k , and $c \in H$. Then, any two bounded Fatou components of f_c have disjoint closures. Moreover, the dynamical root of the characteristic Fatou component of f_c is the landing point of exactly two dynamical rays, each of period $2k$, and these are permuted by f_c^{ok} .*

Proof. Let U_1 and U_2 be two distinct bounded Fatou components of f_c with $\overline{U_1} \cap \overline{U_2} \neq \emptyset$. By taking iterated forward images, we can assume that U_1 and U_2 are periodic. Then the intersection consists of a repelling periodic point x of some period n .

Each periodic (bounded) Fatou component of f_c has period k . Since $x \in \partial U_1 \cap \partial U_2$, it follows that $f_c^{ok}(x) \in \partial U_1 \cap \partial U_2$. But the closures of two distinct bounded Fatou components of an antiholomorphic polynomial may intersect only at a single point. So, $f_c^{ok}(x) = x$. Therefore, the first return map of $\overline{U_1}$ (and of $\overline{U_2}$) fixes x , and x disconnects the Julia set; hence, x is the root of U_1 (as well as of U_2). It follows from [NS03, Corollary 4.2] that $n = k$. But this contradicts the fact that every periodic Fatou component of f_c has exactly one root, and there is exactly one cycle of periodic (bounded) Fatou components of f_c . This proves the first statement of the lemma. The statement about the dynamical rays landing at the characteristic Fatou component of f_c now directly follows from [NS03, Corollary 4.2]. □

In what follows, we delve deep into the behavior of orbit portraits on parabolic arcs and use them to deduce many facts about the parameter rays of the multicorns.

Let H be a hyperbolic component of odd period $k(\neq 1)$ with center c_0 . The characteristic Fatou component U_{c_0} of f_{c_0} has exactly d dynamical co-roots and exactly 1 dynamical root on its boundary (in fact, this is true for every parameter $c \in H$). Let $S = \{\vartheta_1, \vartheta_2, \dots, \vartheta_d\}$ (each angle of period k) be the

set of angles of the dynamical rays that land at the d dynamical co-roots on ∂U_{c_0} , and let $S' = \{\alpha_1, \alpha_2\}$ (each angle of period $2k$) be the set of angles of the dynamical rays that land at the unique dynamical root on ∂U_{c_0} . We will use these terminologies in the next two lemmas, the first one of which shows that the patterns of ray landing at the dynamical co-roots and root remain stable throughout the hyperbolic component.

Lemma 3.3.3. *For any $c \in H$, the set of angles of the dynamical rays of f_c that land at the d dynamical co-roots on ∂U_c (where U_c is the characteristic Fatou component of f_c) is S , and the set of angles of the dynamical rays of f_c that land at the unique dynamical root on ∂U_c is S' .*

Proof. f_{c_0} has a unique super-attracting orbit, and all other periodic orbits of f_{c_0} are repelling. Therefore, the dynamical rays $R_{c_0}(t)$ with $t \in S \cup S'$ land at repelling periodic points. The result now follows from Lemma 3.1.3 since H does not intersect any parabolic parameter or any parameter ray.

We outline an alternative proof of this lemma making use of holomorphic motions. We embed our family $\{f_c\}_{c \in \mathbb{C}}$ in the family of holomorphic polynomials $\{P_{a,b}(z) = (z^d + a)^d + b, a, b \in \mathbb{C}\}$ (recall that $f_c^{\circ 2} = P_{\bar{c},c}$). The family $P_{a,b}$ depends holomorphically on the complex parameters a and b . Let $\tilde{H}$ be the unique hyperbolic component of the family $\{P_{a,b}\}$ that contains the parameter $(\bar{c}_0, c_0)$. Clearly, $\tilde{H}$ contains an open neighborhood of H in $\mathbb{C}^2$. For each $(a, b) \in \tilde{H}$, the Julia set $J(P_{a,b})$ is connected, and hence, the Böttcher coordinate $\varphi_{a,b}$ of $P_{a,b}$ yields a biholomorphism from the basin of infinity $\mathcal{A}_\infty(P_{a,b})$ onto $\hat{\mathbb{C}} \setminus \mathbb{D}$ conjugating $P_{a,b}$ to z^{d^2} . Using this, one can define a holomorphic motion $\Pi : \tilde{H} \times \mathcal{A}_\infty(P_{\bar{c}_0, c_0}) \rightarrow \mathbb{C}$ by requiring $\Pi((a, b), z) = \varphi_{a,b} \circ \varphi_{\bar{c}_0, c_0}^{-1}(z)$ (compare [Zin00, Theorem 6.4] for a similar construction). Using the λ -lemma [MSS83] (note that this holds when the parameter space of the holomorphic motion is biholomorphic to the unit ball in $\mathbb{C}^2$, and this is enough for our purpose), one sees that this holomorphic motion extends to $\overline{\mathcal{A}_\infty(P_{\bar{c}_0, c_0})}$ yielding a topological conjugacy between $P_{\bar{c}_0, c_0}$ on $\overline{\mathcal{A}_\infty(P_{\bar{c}_0, c_0})}$ and $P_{a,b}$ on $\overline{\mathcal{A}_\infty(P_{a,b})}$ such that the conjugacy sends $R_{(\bar{c}_0, c_0)}(t)$ to $R_{(a,b)}(t)$, for all $t \in \mathbb{R}/\mathbb{Z}$ and for all $(a, b) \in \tilde{H}$. Clearly, the property of being a dynamical co-root or root on the boundary of the characteristic Fatou component is also preserved by this topological conjugacy. Since for all $c \in H$, the characteristic Fatou component U_c of f_c has exactly d dynamical co-roots and exactly 1 dynamical root on its boundary, we have thus recovered all the co-roots and roots, and the above argument shows that the set of angles of the dynamical rays of f_c that land at the d dynamical co-roots on ∂U_c is S , and the set of angles of the dynamical rays of f_c that land at the unique dynamical root on ∂U_c is S' . $\square$

By Lemma 3.1.9, ∂H consists of parabolic arcs and cusps of period k .

Lemma 3.3.4. *Let H be a hyperbolic component of odd period k with center c_0 . Let $c \in \partial H$, and the dynamical ray $R_c(\vartheta)$ lands at the characteristic parabolic point of f_c .*

1. *If c lies on a co-root arc on ∂H , then $\vartheta \in S$.*
2. *If c lies on a root arcs on ∂H , then $\vartheta \in S'$.*
3. *If c is a cusp point on ∂H , then $\vartheta \in S \cup S'$.*

Proof. Let z_c be the characteristic parabolic point of f_c . Since the dynamical ray $R_c(\vartheta)$ lands at the k -periodic point z_c , by Lemma 2.1.7, the period of ϑ (under multiplication by $-d$) is either k or $2k$.

1) Since c lies on a co-root arc, z_c is a dynamical co-root of the characteristic Fatou component (of period k) of f_c . By [NS03, Lemma 3.4, Remark 3.5], the period of ϑ must be k , and $R_c(\vartheta)$ is the unique dynamical ray of f_c that land at z_c . Let $c' \in H$ be sufficiently close to c . The characteristic parabolic point z_c of f_c splits into an attracting periodic point of period k and a repelling periodic point (say, $z_{c'}$) of period k of $f_{c'}$. Furthermore, $z_{c'}$ is a dynamical co-root of the characteristic Fatou component of $f_{c'}$. Therefore, the dynamical ray $R_{c'}(\vartheta_i)$ (for some $\vartheta_i \in S$) is the unique dynamical ray of $f_{c'}$ that land at $z_{c'}$. We will now show that the dynamical ray $R_c(\vartheta_i)$ of f_c lands at z_c . This would prove that $\vartheta = \vartheta_i \in S$.

If the dynamical ray $R_c(\vartheta_i)$ lands at a repelling periodic point of f_c , it would continue to land at the continuation of this repelling periodic point for nearby parameters (which would necessarily be different from a dynamical co-root born out of the parabolic point z_c), which is a contradiction. Therefore, $R_c(\vartheta_i)$ must land at a parabolic periodic point of f_c . If $R_c(\vartheta_i)$ lands at a non-characteristic parabolic point of f_c , the orbit separation lemma (Lemma 3.1.14) would supply a partition of the dynamical plane by a pair rational dynamical rays landing at a common point, stable under small perturbations, separating the critical value c from the dynamical ray $R_c(\vartheta_i)$. But for arbitrarily close parameters in H , the dynamical ray at angle ϑ_i lands on the boundary of the characteristic Fatou component, and there does not exist any rational dynamical ray pair separating the ϑ_i ray from the critical value, which is again a contradiction. Thus, the dynamical ray $R_c(\vartheta_i)$ lands at the characteristic parabolic point z_c .

2) This is completely analogous to case (1).

3) Perturbing a cusp point c into H breaks the parabolic cycle into two disjoint repelling cycles (and an attracting one), and if the dynamical ray $R_c(\vartheta)$ lands at the characteristic parabolic point of z_c , then for any nearby parameter $c' \in H$, the dynamical ray $R_{c'}(\vartheta)$ lands at a dynamical co-root or root on the boundary of the characteristic Fatou component of $f_{c'}$. This proves that $\vartheta \in S \cup S'$. The details are similar to case (1). $\square$

The fact that all these possibilities are realized will be proven in Section 3.4.

Let us begin with an elementary fact about the accumulation set of a parameter ray at a periodic (under multiplication by $-d$) angle.

Lemma 3.3.5. *Let ϑ be n -periodic under multiplication by $-d$, $\mathcal{R}_\vartheta^d$ be a parameter ray of $\mathcal{M}_d^*$, and $c \in L_{\mathcal{M}_d^*}(\vartheta)$. Then, f_c has a parabolic cycle of period k dividing n such that the corresponding dynamical ray $R_c(\vartheta)$ lands at the characteristic parabolic point of f_c .*

Proof. This is a classical result for the Mandelbrot set [GM93, Theorem C.7] [Sch00, Proposition 3.1, Proposition 3.2], and can be proved analogously in the antiholomorphic setting. $\square$

Lemma 3.3.6. *For every co-root arc $\mathcal{C}$ of period k , there exists a unique ϑ of period k (under multiplication by $-d$) such that the dynamical ray $R_c(\vartheta)$ lands at the characteristic parabolic point of f_c for each $c \in \mathcal{C}$. In particular, the parabolic orbit portrait is trivial and constant on every co-root arc.*

Proof. By Corollary 3.2.5, there exists a unique hyperbolic component H of odd period k such that $\mathcal{C} \subset \partial H$, and $\mathcal{C}$ does not intersect the boundary of any other hyperbolic component of period k . Let S be the set of angles of the dynamical rays that land at the d dynamical co-roots of the characteristic Fatou component of the center c_0 of H . Therefore, $|S| = d$.

Let $\tilde{c} \in \mathcal{C}$. By the definition of co-root arcs, the parabolic periodic points of $f_{\tilde{c}}$ do not disconnect the Julia set of $f_{\tilde{c}}$; hence every parabolic periodic point of $f_{\tilde{c}}$ is the landing point of exactly one periodic dynamical ray. Let ϑ be the angle of the unique dynamical ray that lands at the characteristic parabolic point of $f_{\tilde{c}}$. By Lemma 3.3.4, $\vartheta \in S$.

In the dynamical plane of $\tilde{c}$, all the dynamical rays at angles in $(S \setminus \{\vartheta\})$ land at repelling periodic points. Hence under small perturbation of $\tilde{c}$ along the co-root arc, the dynamical rays at these angles continue to land at repelling periodic points (by Lemma 3.1.3). Since every parabolic periodic point must be the landing point of at least one (exactly one on the co-root arcs) periodic dynamical ray, the dynamical ray $R_c(\vartheta)$ must land at the characteristic parabolic point of f_c for every parameter $c \in \mathcal{C}$ close to $\tilde{c}$. Therefore, the set $\mathcal{C}' := \{c \in \mathcal{C} : R_c(\vartheta) \text{ lands at the characteristic parabolic point of } f_c\}$ is an open set. Since there are only finitely many choices for ϑ ; $\mathcal{C}$ can be written as the union of finitely many disjoint open sets. It follows from the connectedness of $\mathcal{C}$ that all but one of these open sets are empty. So there exists an angle ϑ such that $R_c(\vartheta)$ lands at the characteristic parabolic point of f_c for every $c \in \mathcal{C}$. Now it trivially follows that the parabolic orbit portrait is constant on $\mathcal{C}$. $\square$

Corollary 3.3.7. *At most one periodic parameter ray can accumulate on a co-root parabolic arc.*

Proof. This follows from Lemma 3.3.5 and Lemma 3.3.6. $\square$

Lemma 3.3.8. *For every root arc $\mathcal{C}$ of period k , there exists a unique pair of angles α_1 and α_2 of period $2k$ (under multiplication by $-d$) such that the dynamical rays $R_c(\alpha_1)$ and $R_c(\alpha_2)$ (and none else) land at the characteristic parabolic point of f_c for each $c \in \mathcal{C}$. In particular, the parabolic orbit portrait is non-trivial and constant on every root arc.*

Proof. By Corollary 3.2.5, there exists a unique hyperbolic component H of odd period k such that $\mathcal{C} \subset \partial H$, and $\mathcal{C}$ does not intersect the boundary of any other hyperbolic component of period k . Let S' be the set of angles of the dynamical rays that land at the unique dynamical root of the characteristic Fatou component of the center c_0 of H . By Lemma 3.3.2, $|S'| = 2$.

Let $\tilde{c} \in \mathcal{C}$. By the definition of root arcs, the parabolic periodic points of $f_{\tilde{c}}$ disconnect the Julia set of $f_{\tilde{c}}$; hence every parabolic periodic point of $f_{\tilde{c}}$ is a dynamical root point. Since this root point has odd period k , it follows from [NS03, Corollary 4.2] that precisely 2 dynamical rays land at this root point. Let α_1 and α_2 be the angles of the dynamical rays that land at the characteristic parabolic point of $f_{\tilde{c}}$. By Lemma 3.3.4, $S' = \{\alpha_1, \alpha_2\}$, and hence, the angles α_1 and α_2 are independent of the choice of $\tilde{c} \in \mathcal{C}$. Therefore, the rays $R_c(\alpha_1)$ and $R_c(\alpha_2)$ (and none else) land at the characteristic parabolic point of f_c for every $c \in \mathcal{C}$. The statement about orbit portraits now follows easily. $\square$

Corollary 3.3.9. *At most two periodic parameter rays can accumulate on a root parabolic arc.*

Proof. This follows from Lemma 3.3.5 and Lemma 3.3.8. $\square$

Lemma 3.3.10 (Where two co-root arcs meet). *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two co-root arcs (of period k) such that the dynamical ray $R_c(\vartheta_1)$ at angle ϑ_1 (respectively $R_c(\vartheta_2)$ at angle ϑ_2) lands at the characteristic parabolic point of f_c for any parameter c on $\mathcal{C}_1$ (respectively on $\mathcal{C}_2$). Let $\tilde{c}$ be the cusp point where these two arcs meet. Then the two dynamical rays $R_{\tilde{c}}(\vartheta_1)$ and $R_{\tilde{c}}(\vartheta_2)$ land at the characteristic parabolic point of $f_{\tilde{c}}$, and no other ray lands there.*

Proof. The dynamical rays $R_{\tilde{c}}(\vartheta_1)$ and $R_{\tilde{c}}(\vartheta_2)$ must land at parabolic points; otherwise by Lemma 3.1.3, they would continue to land at repelling periodic points for nearby parameters contradicting our assumption. To finish the proof, we need to show that these two rays land at the characteristic parabolic point of $f_{\tilde{c}}$: since both these rays have odd period k , this would prove that these are the only rays landing there (Theorem 2.1.8).

If either of the two rays lands at a non-characteristic parabolic point of $f_{\tilde{c}}$, the orbit separation lemma (Lemma 3.1.14) would supply a partition by a pair of rational dynamical rays landing at a common point, stable under small perturbations, separating the critical value from the dynamical ray $R_{\tilde{c}}(\vartheta_1)$

(respectively $R_{\tilde{c}}(\vartheta_2)$). But for parameters on $\mathcal{C}_1$ (respectively $\mathcal{C}_2$) sufficiently close to $\tilde{c}$, the dynamical ray at angle ϑ_1 (respectively ϑ_2) lands at the characteristic parabolic point, and there does not exist any rational dynamical ray pair separating the ϑ_1 (respectively ϑ_2) ray from the critical value: a contradiction! Thus both $R_{\tilde{c}}(\vartheta_1)$ and $R_{\tilde{c}}(\vartheta_2)$ land at the characteristic parabolic point of $f_{\tilde{c}}$. $\square$

Lemma 3.3.11 (Where a co-root and a root arc meet). *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be a co-root and a root arc (of period k) such that the dynamical ray(s) $R_c(\vartheta_1)$ (respectively $R_c(\alpha_1)$ and $R_c(\alpha_2)$) land(s) at the characteristic parabolic point of f_c for each $c \in \mathcal{C}_1$ (respectively $\mathcal{C}_2$). Let $\tilde{c}$ be the cusp point where these two arcs meet. Then the three dynamical rays $R_{\tilde{c}}(\vartheta_1)$, $R_{\tilde{c}}(\alpha_1)$ and $R_{\tilde{c}}(\alpha_2)$ land at the characteristic parabolic point of $f_{\tilde{c}}$, and no other ray lands there.*

Proof. The proof is similar to that of the previous lemma. $\square$

3.4 Boundaries of Odd Period Components

The main goal of this section is to prove Theorem 1.2.2, which describes the structure of the boundaries of hyperbolic components of odd periods of $\mathcal{M}_d^*$.

The basic idea of the proof is to transfer the dynamical co-roots/roots to the parameter plane. Due to the lack of complex analytic parameter dependence, the multiplier map does not extend continuously to the boundary of such a hyperbolic component. Hence, the usual analytic approach is replaced by more combinatorial arguments. Lemma 3.4.3, which can be viewed as a combinatorial rigidity result, lies at the heart of most of our combinatorial arguments.

Recall that a periodic Fatou component is called *characteristic* if it contains the critical value. We will need the concept of *supporting rays*, which will be defined following the ideas in [Poi93, Definition 1.3]. The definition of Poirier is more general, we adapt it to our setting.

Definition 3.4.1 (Supporting Arguments and Rays). *Given a periodic Fatou component U of period n (of f_c) and a point $p \in \partial U$ which is fixed by f_c^n (i.e. so that p is a dynamical root/co-root of U), there are only a finite number of dynamical rays $R_c(\vartheta_1)$, $R_c(\vartheta_2)$, $\dots$, $R_c(\vartheta_v)$ landing at p . These rays divide the plane into v regions. We order the arguments of these rays in counterclockwise cyclic order $\{\vartheta_1, \dots, \vartheta_v\}$, so that U belongs to the region determined by $R_c(\vartheta_1)$ and $R_c(\vartheta_2)$ ($\vartheta_1 = \vartheta_2$ if there is a single ray landing at p). The argument ϑ_1 (respectively the ray $R_c(\vartheta_1)$) is by definition a (left) supporting argument (respectively the (left) supporting ray) of the Fatou component U .*

If f_{c_0} has a periodic critical orbit of odd period k , then the characteristic Fatou component U_{c_0} of f_{c_0} has exactly d dynamical co-roots and exactly 1 dynamical root on its boundary. Let $S = \{\vartheta_1, \vartheta_2, \dots, \vartheta_d\}$ (each angle of

period k) be the set of angles of the dynamical rays that land at the d dynamical co-roots on ∂U_{c_0} , and let $S' = \{\alpha_1, \alpha_2\}$ (each angle of period $2k$) be the set of angles of the dynamical rays that land at the unique dynamical root on ∂U_{c_0} . Furthermore, we have $(-d)^k \alpha_1 = \alpha_2$ and $(-d)^k \alpha_2 = \alpha_1$. Let $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k\}$ be the orbit portrait associated with the periodic orbit of the unique dynamical root on ∂U_{c_0} such that $\mathcal{A}_1 = \{\alpha_1, \alpha_2\}$. It follows that the dynamical rays $R_{c_0}(\alpha_1)$ and $R_{c_0}(\alpha_2)$ along with their common landing point separate the characteristic Fatou component from all other periodic (bounded) Fatou components, and $\{\alpha_1, \alpha_2\}$ are the characteristic angles of $\mathcal{P}$. Without loss of generality, we can assume that the characteristic arc of $\mathcal{P}$ is (α_1, α_2) . According to the definition of supporting rays, each of the dynamical rays $R_{c_0}(\vartheta_1), \dots, R_{c_0}(\vartheta_d)$ and $R_{c_0}(\alpha_1)$ is a (left) supporting ray for the characteristic Fatou component U_{c_0} of f_{c_0} .

To each post-critically finite holomorphic polynomial, one can assign a critical portrait, starting with the choice of preferred (left) supporting rays. A critical portrait is a finite collection of finite subsets of $\mathbb{Q}/\mathbb{Z}$, one subset corresponding to each critical point. This construction is due to Poirier, and we refer the readers to [Poi93, §2] for the details. For the center c_0 of a hyperbolic component of period k of $\mathcal{M}_d^*$, $f_{c_0}^{\circ 2}$ is a post-critically finite polynomial. The choice of a preferred (left) supporting ray (in fact, its angle) on the boundary of the characteristic Fatou component U_{c_0} of f_{c_0} completely determines a critical portrait of $f_{c_0}^2$. Observe that a post-critically finite polynomial may have many critical portraits, depending on the choice of a preferred (left) supporting ray. We will denote a critical portrait associated with the polynomial $f_{c_0}^{\circ 2}$ as $\mathcal{F}_{c_0}$. The choices involved in the construction of a critical portrait adds great flexibility to the following theorem [Poi93, Theorem 2.4], which is a combinatorial way of classifying post-critically finite polynomials. We state the theorem only in the context of the maps $f_c^{\circ 2}$ that arise in our setting.

Theorem 3.4.2. [Poi93, Theorem 2.4] *Let c_0 and c'_0 be the centers of two hyperbolic components of $\mathcal{M}_d^*$. If for some choices of (left) supporting rays, they have the same critical portraits, i.e. if $\mathcal{F}_{c_0} = \mathcal{F}_{c'_0}$, then $c_0 = c'_0$.*

Lemma 3.4.3 (Supporting Rays Determine Dynamics). *1) Suppose that f_{c_1} and f_{c_2} have periodic critical orbits of odd period. Further assume that the dynamical ray $R_{c_i}(\vartheta)$ lands at a dynamical co-root or root of the characteristic Fatou component of f_{c_i} , for $i = 1, 2$. Then, $c_1 = c_2$.*

2) Suppose that f_{c_1} and f_{c_2} have parabolic cycles of odd period, and neither of them is a cusp. Assume further that the critical values of f_{c_1} and f_{c_2} have the same Ecalle height, and there is a ϑ such that the dynamical rays at angle ϑ land at the characteristic parabolic points for both the antiholomorphic polynomials. Then $c_1 = c_2$.

Proof. 1) Under the assumptions of the lemma, ϑ is a supporting argument for the characteristic Fatou component of f_{c_i} , for $i = 1, 2$. The period of ϑ (under

multiplication by $-d$) completely determines the period of the critical cycle of f_{c_1} and of f_{c_2} . Therefore, the periods of the critical cycles of f_{c_1} and f_{c_2} must be equal, say k . This integer k and the (preferred) supporting argument ϑ completely determine a critical portrait for $f_{c_i}^{\circ 2}$, for $i = 1, 2$, and it follows that $f_{c_1}^{\circ 2}$ and $f_{c_2}^{\circ 2}$ have the same critical portraits. By Theorem 3.4.2, $c_1 = c_2$.

2) If f_{c_1} has a parabolic cycle of period k , then c_1 lies on the boundary of a hyperbolic component H (with center c_0 , say) of the same period k . By Lemma 3.3.4, the dynamical ray $R_{c_0}(\vartheta)$ lands at a dynamical co-root or root point of the characteristic Fatou component of f_{c_0} . By part (1), this determines c_0 and thus H uniquely. Therefore, $c_1, c_2 \in \partial H$. Since, none of them is a cusp, f_{c_1} and f_{c_2} have the same rational lamination.

Since c_1 and c_2 have identical Ecalle height, their Fatou coordinates provide a conformal conjugacy $\psi^1 : f_{c_1} \sim f_{c_2}$ in the immediate parabolic basins, hence from the interior of the filled-in Julia set $K(f_{c_1})$ to the interior of the filled-in Julia set $K(f_{c_2})$. Since the Julia sets are locally connected, so are the closures of every bounded Fatou component. Therefore, ψ^1 extends homeomorphically to the boundary of every bounded Fatou component and conjugates f_{c_1} to f_{c_2} .

On the other hand, by the normalized Riemann maps φ_{c_i} of the basins of infinity $\mathcal{A}_\infty(f_{c_i})$ constructed in [NS03], we define a conformal conjugacy $\psi^2 = \varphi_{c_2}^{-1} \circ \varphi_{c_1} : f_{c_1} \sim f_{c_2}$ from the basin of infinity $\mathcal{A}_\infty(f_{c_1})$ of f_{c_1} to the basin of infinity $\mathcal{A}_\infty(f_{c_2})$ of f_{c_2} . Since the parabolic Julia set $J(f_{c_i})$ is locally connected, the inverse of the Riemann maps $\varphi_{c_i}^{-1}$ extend continuously to $\overline{\mathbb{D}}$. Since f_{c_1} and f_{c_2} have the same rational lamination, the conjugacy ψ^2 extends to a homeomorphism of $\mathcal{A}_\infty(f_{c_1})$ onto $\mathcal{A}_\infty(f_{c_2})$ such that it maps the parabolic periodic points (and their inverse orbits) of f_{c_1} to those of f_{c_2} (in particular, the characteristic parabolic point of f_{c_1} maps to that of f_{c_2}).

It easily follows from our construction that ψ^1 and ψ^2 agree on a dense subset of their common domains of definition (the union of the boundaries of the bounded Fatou components); namely, on the parabolic cycle and their iterated pre-images. Thus, ψ^1 and ψ^2 coincide everywhere on their common domains of definition and define a homeomorphism $\psi : \mathbb{C} \rightarrow \mathbb{C}$, which is a topological conjugacy between f_{c_1} and f_{c_2} and conformal outside $J(f_{c_1})$.

We will construct a conformal conjugacy by a similar argument as in [HS14, Lemma 5.8]. Consider the equipotential E of f_{c_1} at some positive potential, and let $E_1, \dots, E_k$ be piecewise analytic simple closed curves, one in each bounded periodic Fatou component of f_{c_1} , that surround the post-critical set in their Fatou components and that intersect the boundary of their Fatou components in one point, which is on the parabolic cycle. Let V_0 be the domain bounded on the outside by E and on the inside by the E_i 's. Then there is a quasi-conformal homeomorphism $\psi_1 : \mathbb{C} \rightarrow \mathbb{C}$ with $\psi_1 = \psi$ on $\mathbb{C} \setminus V_0$ (i.e. the homeomorphism ψ is modified on V_0 so as to become quasi-conformal, possibly giving up on the condition that ψ_1 is a conjugation on V_0).

Now construct a sequence of quasi-conformal homeomorphisms $\psi_n : \mathbb{C} \rightarrow \mathbb{C}$

as a sequence of pullbacks, satisfying $f_{c_2} \circ \psi_{n+1} = \psi_n \circ f_{c_1}$ with $\psi_{n+1} = \psi_n$ on $f_{c_1}^{-n}(\mathbb{C} \setminus V_0)$: since the initial conjugacy respects the critical orbits, this construction is possible, and all ψ_n satisfy the same bounds on the quasi-conformal dilatation as ψ_1 . Therefore, the sequence $\{\psi_n\}$ converges to a topological conjugation ψ_∞ . Moreover, the supports of the quasi-conformal dilatations of $\{\psi_n\}$ shrink to the Julia set, which has measure zero. By compactness of the space of quasi-conformal maps with a given dilatation, ψ_∞ is a quasi-conformal conjugacy between f_{c_1} and f_{c_2} , that is conformal almost everywhere. Hence, ψ_∞ is a conformal conjugacy between f_{c_1} and f_{c_2} .

Now f_{c_1} is conformally, hence affinely conjugate to f_{c_2} . However, since the Riemann maps φ_{c_i} are normalized, hence tangent to the identity near ∞ , so is ψ^2 , and hence the conformal conjugation ψ_∞ . This implies $f_{c_1} = f_{c_2}$. $\square$

Using this lemma, we can give an upper bound on the number of parabolic arcs on the boundary of a hyperbolic component of odd period.

Lemma 3.4.4 (Upper Bound on the Number of Parabolic Arcs). *Every hyperbolic component of odd period, different from 1, has at most d co-root arcs and at most 1 root arc on its boundary. The hyperbolic component of period 1 has exactly $d + 1$ co-root arcs on its boundary.*

Proof. Let H be a hyperbolic component of odd period $k \neq 1$. We first give an upper bound on the number of co-root arcs on ∂H .

Let $S = \{\vartheta_1, \vartheta_2, \dots, \vartheta_d\}$ be the set of angles of the dynamical rays that land at the d dynamical co-roots of the characteristic Fatou component of the center of H . Recall that by Lemma 3.3.6 and its proof, for every co-root arc $\mathcal{C} \subset \partial H$, there exists a *unique* ϑ of period k (under multiplication by $-d$) such that $R_c(\vartheta)$ is the only dynamical ray landing at the characteristic parabolic point of f_c , for each $c \in \mathcal{C}$. Moreover, $\vartheta \in S$. Now suppose that there are more than d co-root arcs on ∂H . Then there would exist two distinct co-root arcs $\mathcal{C}_1$ and $\mathcal{C}_2$ contained in ∂H such that the dynamical ray $R_c(\vartheta)$ at angle ϑ (say) is the unique ray landing at the characteristic parabolic point of f_c for all $c \in \mathcal{C}_1 \cup \mathcal{C}_2$. For $i = 1, 2$, let c_i be the parameter on $\mathcal{C}_i$ such that the critical Ecalle height of c_i is 0. By Lemma 3.2.4, $c_1 \neq c_2$. This contradicts part (2) of Lemma 3.4.3.

Now we turn our attention to the root arcs. Let $S' = \{\alpha_1, \alpha_2\}$ (both angles of period $2k$) be the set of angles of the dynamical rays that land at the unique dynamical root of the characteristic Fatou component of the center of H (see Lemma 3.3.2). By Lemma 3.3.8 and its proof, for every root arc $\mathcal{C} \subset \partial H$, the dynamical rays $R_c(\alpha_1)$ and $R_c(\alpha_2)$ (and none else) land at the characteristic parabolic point of f_c for each $c \in \mathcal{C}$. Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are two distinct root arcs contained in ∂H . For $i = 1, 2$, let c_i be the parameter on $\mathcal{C}_i$ such that the critical Ecalle height of c_i is 0. Then, the dynamical ray $R_{c_i}(\alpha_1)$ would land at the characteristic parabolic point of f_{c_i} for $i = 1, 2$. By Lemma 3.4.3,

$c_1 = c_2$. But this contradicts Lemma 3.2.4, which states that two distinct parabolic arcs never intersect.

Finally, we deal with the hyperbolic component of period 1. The characteristic Fatou component (which is the unique bounded Fatou component) of the center of the period 1 hyperbolic component has exactly $d + 1$ co-roots and no root. Arguing as above, one easily sees that there are at most $d + 1$ co-root arcs on the boundary of this hyperbolic component. By [NS03, Corollary 5.10], every hyperbolic component of odd period k has a unique center c_0 such that the unique critical point 0 of f_{c_0} has a k -periodic orbit. For $k = 1$, the only such unicritical antiholomorphic polynomial is $\bar{z}^d$. This proves that there is exactly one hyperbolic component H_0 of period 1 in $\mathcal{M}_d^*$. Now recall that $\mathcal{M}_d^*$ has a $(d + 1)$ -fold rotational symmetry. More precisely, if ω is a primitive $(d + 1)$ -th root of unity, then multiplication by ω generates a cyclic group (of order $d + 1$) of symmetries of $\mathcal{M}_d^*$. In particular, multiplication by ω yields a $(d + 1)$ -fold rotational symmetry of the unique hyperbolic component H_0 of period 1 of $\mathcal{M}_d^*$. Therefore, if $\mathcal{C}$ is a parabolic arc on ∂H_0 , then $\omega\mathcal{C}, \omega^2\mathcal{C}, \dots, \omega^d\mathcal{C}$ are also distinct parabolic arcs on ∂H_0 . This shows the existence of $d + 1$ parabolic arcs on ∂H_0 . Since there are at most $d + 1$ of them, we conclude that there are exactly $d + 1$ co-root arcs on the boundary of the period 1 hyperbolic component. $\square$

Here is another elementary application of Lemma 3.4.3.

Lemma 3.4.5. *Any two root arcs have disjoint closures.*

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two distinct root arcs. By Lemma 3.3.8, there exist two pairs of angles $\{\alpha_1, \alpha_2\}$ and $\{\alpha'_1, \alpha'_2\}$ such that the dynamical rays $R_c(\alpha_1)$ and $R_c(\alpha_2)$ (respectively $R_c(\alpha'_1)$ and $R_c(\alpha'_2)$) land at the characteristic parabolic point of f_c for each $c \in \mathcal{C}_1$ (respectively for each $c \in \mathcal{C}_2$). Arguing as in Lemma 3.4.4 (using Lemma 3.4.3), one sees that $\{\alpha_1, \alpha_2\} \cap \{\alpha'_1, \alpha'_2\} = \emptyset$.

By Lemma 3.2.4, $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. Let $\tilde{c} \in \overline{\mathcal{C}_1} \cap \overline{\mathcal{C}_2}$. Arguing as in Lemma 3.3.10, we see that four distinct dynamical rays $R_{\tilde{c}}(\alpha_1)$, $R_{\tilde{c}}(\alpha_2)$, $R_{\tilde{c}}(\alpha'_1)$, and $R_{\tilde{c}}(\alpha'_2)$ (each of period $2k$) land at the characteristic parabolic point of $f_{\tilde{c}}$. This contradicts Lemma 2.1.13. Hence, $\overline{\mathcal{C}_1} \cap \overline{\mathcal{C}_2} = \emptyset$. $\square$

The next application of Lemma 3.4.3 tells us where a parameter ray of odd period must accumulate.

Lemma 3.4.6. *Let ϑ be the angle (of period k) of the dynamical ray that lands at a dynamical co-root of the characteristic Fatou component of the center c_0 of the hyperbolic component H of odd period k . Then the parameter ray $\mathcal{R}_\vartheta^d$ either accumulates on the closure of a single co-root arc of H or lands at a cusp on ∂H .*

Proof. Every accumulation point c of $\mathcal{R}_\vartheta^d$ must have a parabolic cycle of period k , and the dynamical ray $R_c(\vartheta)$ must land at the characteristic parabolic point

of f_c (by Lemma 3.3.5 and Lemma 2.1.7). By Theorem 3.1.1, c lies on the boundary of a hyperbolic component H' of period k . By Lemma 3.3.4, the center c'_0 of H' has the property that the dynamical ray $R_{c'_0}(\vartheta)$ lands at a dynamical co-root of the characteristic Fatou component of $f_{c'_0}$. Now Lemma 3.4.3 implies that $c_0 = c'_0$; i.e. $H = H'$. Therefore, the accumulation set of $\mathcal{R}_\vartheta^d$ is contained in ∂H .

Since ϑ has odd period k , any accumulation point of $\mathcal{R}_\vartheta^d$ must lie on a co-root arc or a cusp on ∂H (compare Lemma 3.3.8). By the proof of Lemma 3.4.4, there is at most one co-root arc $\mathcal{C}$ on the boundary of ∂H such that the dynamical ray $R_c(\vartheta)$ lands at the characteristic parabolic point of f_c for each $c \in \mathcal{C}$. Since $L_{\mathcal{M}^*}(\vartheta)$ is connected, this shows that $\mathcal{R}_\vartheta^d$ either lands at some cusp point on ∂H or accumulates on the closure of a single co-root arc on ∂H . $\square$

It follows from Lemma 3.3.2 that if c_0 is the center of a hyperbolic component H of odd period k , then exactly two dynamical rays $R_{c_0}(\alpha_1)$ and $R_{c_0}(\alpha_2)$ (say) land at the dynamical root z_{c_0} of the characteristic Fatou component U_{c_0} of f_{c_0} , and both these rays have period $2k$ (under multiplication by $-d$). Further, we have $(-d)^k \alpha_1 = \alpha_2$ and $(-d)^k \alpha_2 = \alpha_1$. Let $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k\}$ be the orbit portrait associated with the periodic orbit of z_{c_0} such that $\mathcal{A}_1 = \{\alpha_1, \alpha_2\}$. It follows that the dynamical rays $R_{c_0}(\alpha_1)$ and $R_{c_0}(\alpha_2)$ along with their common landing point separate the characteristic Fatou component from all other periodic (bounded) Fatou components, and $\{\alpha_1, \alpha_2\}$ are the characteristic angles of $\mathcal{P}$. Without loss of generality, we can assume that the characteristic arc of $\mathcal{P}$ is (α_1, α_2) . This fact as well as the terminologies of this paragraph will be used in the following lemma.

Lemma 3.4.7. *Let α_1 and α_2 be the angles of the (only) dynamical rays that land at the dynamical root of the characteristic Fatou component of the center of a hyperbolic component H of odd period. Then the parameter rays $\mathcal{R}_{\alpha_1}^d$ and $\mathcal{R}_{\alpha_2}^d$ either accumulate on the closure of a common root arc of H or land at a common cusp point on ∂H .*

Proof. Let the period of H be k . The common period of α_1 and α_2 under multiplication by $-d$ is $2k$. Any accumulation point of these two parameter rays is either a parabolic parameter of odd period k or a parabolic parameter of even period r with $r|2k$ such that the corresponding dynamical ray of period $2k$ lands at the characteristic parabolic point in the dynamical plane of that parameter (by Lemma 3.3.5 and Lemma 2.1.7). Define the set F to be the union of the closure of the finitely many root arcs and cusp points of period k , and the finitely many parabolic parameters of even parabolic period and of ray period $2k$ (this is precisely the set of all parameters c for which f_c has a parabolic cycle such that a dynamical ray $R_c(t)$ of period $2k$ may land at the parabolic cycle of f_c). It follows that the set of accumulation points of $\mathcal{R}_{\alpha_1}^d$ and $\mathcal{R}_{\alpha_2}^d$ is contained in F . We define $\mathcal{AP} := \mathcal{A}_1 \cup \dots \cup \mathcal{A}_k$.

Consider the connected components U_i of

$$\mathbb{C} \setminus \left(\bigcup_{t \in \mathcal{AP}} \mathcal{R}_t^d \cup F \right).$$

There are only finitely many components U_i , and they are open and unbounded. Observe that if we define $A := \{\alpha_1, \alpha_2\}$, then the set $\tilde{A} := \bigcup_{j \geq 0} (-d)^j A$ coincides with the set $\mathcal{AP}$.

The rest of the proof is similar to that of [Mil00b, Theorem 3.1]. It follows from the discussion before this lemma that (α_1, α_2) is the characteristic arc of $\mathcal{P}$. Let U_1 be the component which contains all parameters c outside $\mathcal{M}_d^*$ with external angle $t(c) \in (\alpha_1, \alpha_2)$ (there is such a component as (α_1, α_2) does not contain any other angle of $\mathcal{AP}$). U_1 must have the two parameter rays $\mathcal{R}_{\alpha_1}^d$ and $\mathcal{R}_{\alpha_2}^d$ on its boundary. Pick $c' \in U_1 \setminus \mathcal{M}_d^*$ such that $t(c') \in (\alpha_1, \alpha_2)$. By Lemma 2.2.5, the dynamical rays $R_{c'}(\alpha_1)$ and $R_{c'}(\alpha_2)$ land at a common repelling periodic point of $f_{c'}$. Since $U_1 \subset \mathbb{C} \setminus \left(\bigcup_{t \in \mathcal{AP}} \mathcal{R}_t^d \cup F \right)$, it follows from Lemma 3.1.3 that $R_c(\alpha_1)$ and $R_c(\alpha_2)$ land at a common repelling periodic point of f_c for all c in U_1 .

If the two parameter rays $\mathcal{R}_{\alpha_1}^d$ and $\mathcal{R}_{\alpha_2}^d$ do not land at the same point or do not accumulate on the closure of the same root arc (recall that every hyperbolic component of odd period has at most one root arc, and any two root arcs have disjoint closures), then U_1 would contain parameters c outside $\mathcal{M}_d^*$ with $t(c) \notin (\alpha_1, \alpha_2)$. But it follows from the remark at the end of the proof of Theorem 2.2.2 that for such a parameter c , the dynamical rays $R_c(\alpha_1)$ and $R_c(\alpha_2)$ can not land at a common point, which is a contradiction to the stability of the ray landing pattern throughout U_i . Hence, the parameter rays $\mathcal{R}_{\alpha_1}^d$ and $\mathcal{R}_{\alpha_2}^d$ must either land at the same parabolic parameter of even (parabolic) period and of ray period $2k$, or accumulate on the closure of the same root arc/cusp of $\mathcal{M}_d^*$.

Let us assume that $\mathcal{R}_{\alpha_1}^d$ and $\mathcal{R}_{\alpha_2}^d$ land at a common parabolic parameter c of even (parabolic) period, and we will arrive at a contradiction. By Lemma 3.3.5, the dynamical rays $R_c(\alpha_1)$ and $R_c(\alpha_2)$ of f_c land at the characteristic parabolic point z_c of even period of f_c . But, $f_c^{\circ k}(R_c(\alpha_1)) = R_c((-d)^k \alpha_1) = R_c(\alpha_2)$. As $R_c(\alpha_1)$ lands at z_c , it follows by continuity that $R_c(\alpha_2)$ lands at $f_c^{\circ k}(z_c)$. Thus, $f_c^{\circ k}(z_c) = z_c$. This implies that the period of the parabolic point z_c (under f_c) divides the odd integer k , a contradiction.

Therefore, the parameter rays $\mathcal{R}_{\alpha_1}^d$ and $\mathcal{R}_{\alpha_2}^d$ either accumulate on the closure of a common root arc of H' or land at a common cusp point on $\partial H'$, for some hyperbolic component H' of odd period k . Then, by Lemma 3.3.4, the center c'_0 (say) of H' has the property that the dynamical rays $R_{c'_0}(\alpha_1)$ and $R_{c'_0}(\alpha_2)$ land at the dynamical root $z'_{c'_0}$ of the characteristic Fatou component $U'_{c'_0}$ of $f_{c'_0}$. By Lemma 3.4.3, $c_0 = c'_0$; i.e. $H = H'$. This proves the lemma. $\square$

Lemma 3.4.8. *A cusp point can be contained in the accumulation set of at most three parameter rays at periodic angles.*

Proof. Let c be a cusp point of odd period k . If c is contained in the accumulation set of some periodic parameter ray $\mathcal{R}_\vartheta^d$, then by Lemma 3.3.5, the dynamical ray $R_c(\vartheta)$ lands at the characteristic parabolic point of f_c . The characteristic parabolic point of f_c has odd period k . By Lemma 2.1.13, at most three periodic dynamical rays can land at a periodic point of odd period. This shows that c can be contained in the accumulation set of at most three parameter rays. $\square$

Lemma 3.4.9 (Lower Bound on the Number of Parabolic Arcs). *Every hyperbolic component H of odd period has at least 3 parabolic arcs on its boundary.*

Proof. For the hyperbolic component of period 1, this follows from Lemma 3.4.4. So we only need to work with the case when the period of H is different from 1. If c_0 is the center of H , then $f_{c_0}^{\circ k}$ has exactly $d + 1$ fixed points on the boundary of the characteristic Fatou component of f_{c_0} . Exactly d of them are the landing points of a single dynamical ray, and the remaining one is the landing point of precisely two rays of period $2k$ (by Lemma 3.3.1 and Lemma 3.3.2); this yields a total of $d + 2$ (≥ 4) rays. Call the set of all parameter rays at these $d + 2$ angles Ω . Then, $|\Omega| \geq 4$. The members of Ω must accumulate on ∂H , by Lemma 3.4.6 and 3.4.7.

Let $\mathcal{C}$ be a parabolic arc on ∂H with the cusp point $\tilde{c}$ at one of its ends. Assume that the dynamical ray $R_c(t)$ lands at the characteristic parabolic point of f_c for all $c \in \mathcal{C}$. Then by Lemma 3.3.10 and Lemma 3.3.11, the dynamical ray $R_{\tilde{c}}(t)$ lands at the characteristic parabolic point of $f_{\tilde{c}}$. Therefore, by Lemma 3.3.5, the set of element(s) of Ω that can possibly accumulate on a parabolic arc $\mathcal{C}$ on ∂H is a subset of the set of element(s) of Ω that can possibly accumulate on the cusp points on the ends of the parabolic arc $\mathcal{C}$. At most 3 members of Ω can accumulate at any cusp (by Lemma 3.4.8), at most 2 on any parabolic arc (Corollaries 3.3.7, 3.3.9).

If ∂H contained only a single parabolic arc, then it would contain only a single cusp, and hence, at most 3 members of Ω can accumulate on ∂H . If ∂H had exactly two parabolic arcs on its boundary, then these would either be both co-root arcs, or a root and a co-root arc (by Lemma 3.4.4). In both cases, it is again easy to see that at most 3 members of Ω can accumulate on ∂H . Since $|\Omega| \geq 4$, it follows that ∂H must contain at least 3 parabolic arcs. $\square$

Corollary 3.4.10. *For every hyperbolic component H of odd period k , the boundary ∂H is a simple closed curve.*

Proof. By Lemma 3.1.9, ∂H consists of parabolic arcs and cusps of period k . By Corollary 3.2.5, if a parabolic arc $\mathcal{C}$ intersects ∂H , then $\mathcal{C} \subset \partial H$. Each arc

limits on both ends at cusps, by Lemma 3.2.3. The number of parabolic arcs and cusps on ∂H is finite, by Lemma 3.4.4 and Lemma 3.1.10.

Since H is homeomorphic to $\mathbb{D}$ (by [NS03, §5], the above results imply that ∂H must be a closed curve traversing finitely many parabolic arcs and cusps. Therefore, we can parametrize ∂H by a continuous surjection $\gamma : [0, 1] \rightarrow \partial H$, such that $\gamma(0) = \gamma(1)$. We claim that ∂H is a simple closed curve; if this was not the case, then we would have $t_1 \in [0, 1]$ and $t_2 \in (0, 1)$ with $\gamma(t_1) = \gamma(t_2)$. The rest of the proof will work towards finding a contradiction.

Recall that every parabolic arc is a simple arc (by Theorem 3.2.1), and two distinct parabolic arcs are disjoint (by Lemma 3.2.4). Therefore, the parameter $\gamma(t_1)$ must be a cusp point. This implies that there are at least three different parabolic arcs (Lemma 3.4.9 asserts that ∂H must contain at least 3 parabolic arcs) meeting at the cusp point $\gamma(t_1)$. Let these three parabolic arcs be $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_3$. By Lemma 3.4.4, at most one of them can be a root arc. We consider two cases.

Case 1 ($\mathcal{C}_1$ is a root arc, and $\mathcal{C}_2, \mathcal{C}_3$ are co-root arcs). Arguing as in Lemma 3.3.8 and Lemma 3.3.6, we see that there are distinct angles $\alpha_1, \alpha_2, \vartheta_1$ and ϑ_2 such that the dynamical rays $R_c(\alpha_1)$ and $R_c(\alpha_2)$ land at the characteristic parabolic point of f_c for every $c \in \mathcal{C}_1$, and the dynamical ray $R_c(\vartheta_i)$ lands at the characteristic parabolic point of f_c for every $c \in \mathcal{C}_i$, for $i = 2, 3$. Furthermore, α_1 and α_2 have period $2k$, and ϑ_1 and ϑ_2 have period k . The fact that ϑ_1 and ϑ_2 are different, follows from the proof of Lemma 3.4.4. Now, arguing as in Lemma 3.3.10 and Lemma 3.3.11, we deduce that four dynamical rays $R_{\gamma(t_1)}(\alpha_1)$, $R_{\gamma(t_1)}(\alpha_2)$, $R_{\gamma(t_1)}(\vartheta_1)$ and $R_{\gamma(t_1)}(\vartheta_2)$ land at the characteristic parabolic point (of odd period k) of $f_{\gamma(t_1)}$. But this contradicts Lemma 2.1.13, which states that at most three periodic dynamical rays can land at a periodic point of odd period.

Case 2 (Each $\mathcal{C}_i$, $i = 1, 2, 3$, is a co-root arc). Arguing as in Lemma 3.3.6, we see that there are distinct angles ϑ_1, ϑ_2 and ϑ_3 such that the dynamical ray $R_c(\vartheta_i)$ lands at the characteristic parabolic point of f_c for every $c \in \mathcal{C}_i$, for $i = 1, 2, 3$. Furthermore, each ϑ_i has period k . The fact that all the ϑ_i are different, follows from the proof of Lemma 3.4.4. Now, arguing as in Lemma 3.3.10, we deduce that three dynamical rays $R_{\gamma(t_1)}(\vartheta_1)$, $R_{\gamma(t_1)}(\vartheta_2)$ and $R_{\gamma(t_1)}(\vartheta_3)$ of odd period k land at the characteristic parabolic point (of odd period k) of $f_{\gamma(t_1)}$. But this contradicts Theorem 2.1.8.

Hence, ∂H is a simple closed curve. $\square$

Having proved that the boundary of every hyperbolic component of odd period is a simple closed curve, we have already taken a fundamental step towards the proof of Theorem 1.2.2. In the next few lemmas, we draw some important conclusions about the parabolic arcs and cusp points, which will allow us to give the exact count of parabolic arcs and cusps on the boundary of a hyperbolic component of odd period.

Corollary 3.4.11 (Closure of Arcs Meet Boundary). *The closure of every parabolic arc $\mathcal{C}$ intersects $\partial\mathcal{M}_d^*$.*

Proof. We give a proof for the co-root arcs, the case of the root arcs is similar. Note that any co-root arc $\mathcal{C}$ lies on the boundary of some hyperbolic component H of odd period k (by Corollary 3.2.5). Let ϑ be the external angle (of period k) of the unique dynamical ray landing at the characteristic parabolic point of f_c for any $c \in \mathcal{C}$ (see Lemma 3.3.6). By Lemma 3.3.4, the dynamical ray $R_{c_0}(\vartheta)$ lands at a dynamical co-root of the characteristic Fatou component of the center c_0 of the hyperbolic component H . Hence, by Lemma 3.4.6, the accumulation set $L_{\mathcal{M}_d^*}(\vartheta)$ of the parameter ray $\mathcal{R}_\vartheta^d$ must be contained in ∂H .

If $\tilde{c} \in L_{\mathcal{M}_d^*}(\vartheta)$ is not a cusp, then $\tilde{c}$ is on the given parabolic arc $\mathcal{C}$ (by Lemma 3.3.5 and Lemma 3.4.3). If $\tilde{c} \in L_{\mathcal{M}_d^*}(\vartheta)$ is a cusp, then there must be two parabolic arcs terminating at $\tilde{c}$ (since ∂H is a simple closed curve consisting of finitely many parabolic arcs and cusp points). Let these two parabolic arcs be $\mathcal{C}_1$ and $\mathcal{C}_2$. By Lemma 3.3.10 and Lemma 3.3.11, one of these two arcs, say $\mathcal{C}_1$, has the property that the dynamical ray $R_c(\vartheta)$ lands at the characteristic parabolic point of f_c for each $c \in \mathcal{C}_1$. Now it follows from Lemma 3.4.3 that $\mathcal{C} = \mathcal{C}_1$. In either case, $\overline{\mathcal{C}} \cap L_{\mathcal{M}_d^*}(\vartheta) \neq \emptyset$; i.e. the closure of the parabolic arc $\mathcal{C}$ intersects $\partial\mathcal{M}_d^*$. $\square$

Corollary 3.4.12 (Bifurcations From Odd Periods). *Every cusp point having a parabolic cycle of odd period k sits on the boundary of a hyperbolic component of period $2k$.*

Proof. Let $\tilde{c}$ be a cusp point of period k . By Theorem 3.1.1, there exists a hyperbolic component H of period k such that $\tilde{c} \in \partial H$. Recall that ∂H is a simple closed curve (Corollary 3.4.10). Moreover, by Lemma 3.1.9, ∂H consists only of parabolic arcs and cusp points of period k (more precisely, for each $c \in \partial H$, f_c has a parabolic cycle of period k of multiplicity 1 or 2). But since there are only finitely many cusp points of period k (Lemma 3.1.10), there exists a parabolic arc $\mathcal{C}$ of period k (on ∂H) limiting at $\tilde{c}$. Now, by Theorem 1.1.20, each end of $\mathcal{C}$ intersects the boundary of a hyperbolic component of period $2k$. Let H' be the hyperbolic component of period $2k$ bifurcating from that end of $\mathcal{C}$ which limits at $\tilde{c}$. Since $\tilde{c}$ is a limit point of $\mathcal{C}$, it follows that $\tilde{c} \in \partial H'$. $\square$

Lemma 3.4.13 (Cusps in Interior). *Every parabolic cusp is in the interior of $\mathcal{M}_d^*$.*

Proof. If $\tilde{c}$ is a parabolic cusp, then it has a parabolic cycle of odd period k , so it is simultaneously on the boundary of a period k component W by Theorem 3.1.1, and of a period $2k$ component W' by Corollary 3.4.12. The dynamical rays landing together at the dynamical root of the characteristic Fatou component of the centers of W and W' are uniquely determined by the

dynamical rays landing at the characteristic parabolic point of $f_{\tilde{c}}$, by Lemma 3.3.4. This determines the parameters at the centers of W and W' uniquely, so W and W' are the only components with $\tilde{c}$ on its boundary.

The boundary of W is a simple closed curve by Corollary 3.4.10. Let c_1 and c_2 be two points on the two parabolic arcs $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively that terminate at $\tilde{c}$, and sufficiently close to $\tilde{c}$ so that $c_1, c_2 \in \partial W \cap \partial W'$. Connect c_1 to c_2 by a simple arc γ' within W' , and by another simple arc γ'' within W (this is possible since hyperbolic components are homeomorphic to $\mathbb{D}$, by [NS03, §5]). Then the curve $\gamma := \gamma' \cup \gamma''$ is a simple closed curve. We now consider two cases.

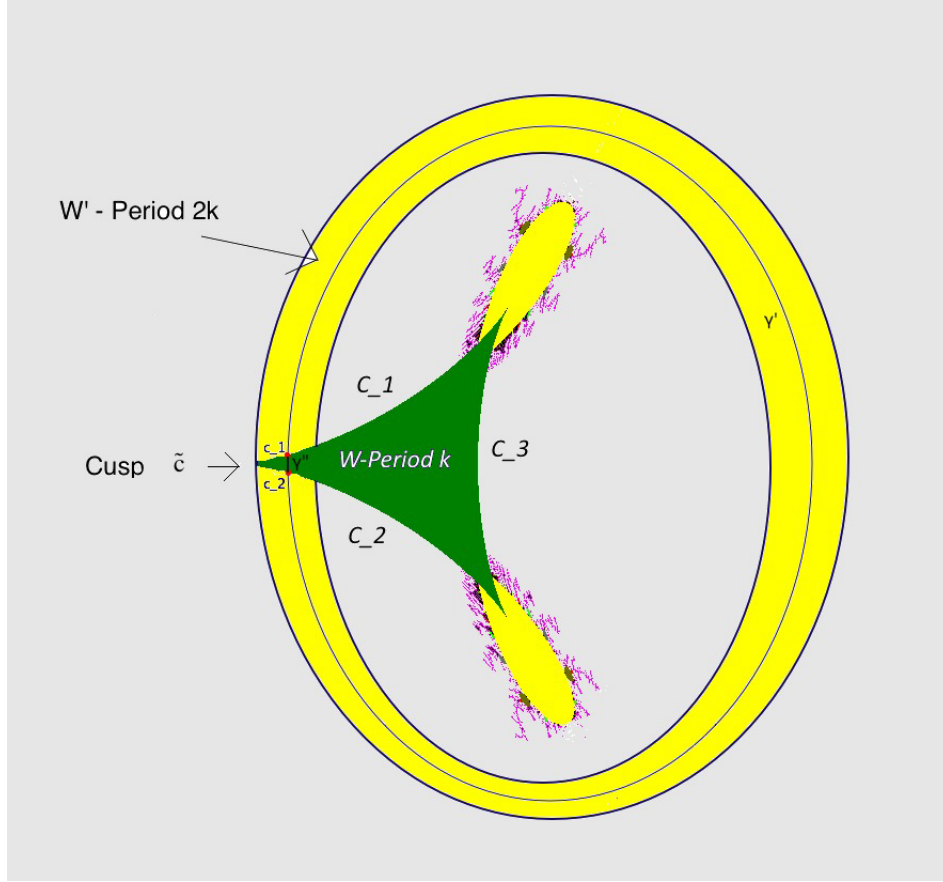


Figure 3.1: If the simple closed curve $\gamma = \gamma' \cup \gamma''$ does not wind around $\tilde{c}$, then the closure of the parabolic arc $\mathcal{C}_3$ would be contained in the unique bounded component U of $\mathbb{C} \setminus \gamma$, proving that the closure of $\mathcal{C}_3$ would lie in the interior of $\mathcal{M}_d^*$.

Case 1 (The curve γ winds around $\tilde{c}$). Note that $\gamma \subset \mathcal{M}_d^*$. Since $\mathcal{M}_d^*$ is a full continuum [Nak93a], the unique bounded component U of $\mathbb{C} \setminus \gamma$ is contained

in $\text{int}(\mathcal{M}_d^*)$. By our assumption, $\tilde{c} \in U$. This shows that $\tilde{c} \in \text{int}(\mathcal{M}_d^*)$.

Case 2 (The curve γ does not wind around $\tilde{c}$). If γ does not wind around $\tilde{c}$, then $\partial W \cap \partial \mathcal{M}_d^* = \{\tilde{c}\}$. Since ∂W contains at least 3 parabolic arcs (by Lemma 3.4.9), this implies that there exists a third parabolic arc $\mathcal{C}_3 \subset \partial W$ such that γ winds around every point $c \in \overline{\mathcal{C}_3}$ (compare Figure 3.1). Since $\mathcal{M}_d^*$ is a full continuum [Nak93a], the unique bounded component U of $\mathbb{C} \setminus \gamma$ is contained in $\text{int}(\mathcal{M}_d^*)$. Therefore, $\overline{\mathcal{C}_3} \subset U \subset \text{int}(\mathcal{M}_d^*)$. But $\overline{\mathcal{C}_3}$ must intersect $\partial \mathcal{M}_d^*$ (by Corollary 3.4.11), which is a contradiction to the assumption of case (2).

This shows that every cusp point $\tilde{c}$ is contained in the interior of $\mathcal{M}_d^*$. $\square$

Corollary 3.4.14 (Arcs Meet Boundary). *Every parabolic arc intersects $\partial \mathcal{M}_d^*$.*

Proof. This is clear from Corollary 3.4.11 and Lemma 3.4.13. $\square$

Corollary 3.4.15. *Every parameter ray at an angle ϑ of odd period k lands/accumulates on a sub-arc of a single co-root parabolic arc of period k .*

Lemma 3.4.16. *Let a rational parameter ray at an odd-periodic angle ϑ lands/accumulates on a sub-arc of the parabolic arc $\mathcal{C}$. Then the impression of this parameter ray contains $\mathcal{C} \cap \partial \mathcal{M}_d^*$.*

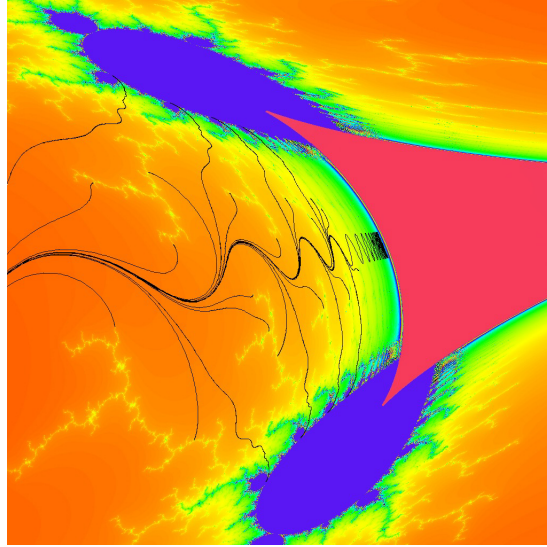


Figure 3.2: The parameter ray at an odd-periodic angle ϑ accumulates on the parabolic arc, and no other ray accumulates there; all rays at nearby angles swerve to the left or to the right.

Proof. By Corollary 3.3.7, $\mathcal{R}_\vartheta^d$ is the only parameter ray at a periodic angle (under multiplication by $-d$) that can accumulate on $\mathcal{C}$. Let $t \in \mathbb{R}/\mathbb{Z}$ (say $t > \vartheta$) be such that $L_{\mathcal{M}_d^*}(t) \cap \mathcal{C} \neq \emptyset$. Then there exists a periodic angle (under multiplication by $-d$) $t' \in (\vartheta, t)$. It follows that $L_{\mathcal{M}_d^*}(t') \cap \mathcal{C} \neq \emptyset$; a contradiction to Corollary 3.3.7. This shows that $\mathcal{R}_\vartheta^d$ is the only parameter ray whose accumulation set intersects $\mathcal{C}$. It is now easy to see that every neighborhood of a point in $\mathcal{C} \cap \partial\mathcal{M}_d^*$ intersects some parameter ray at angle sufficiently close to ϑ (compare Figure 3.2), which proves the claim. $\square$

Proof of Theorem 1.2.2. For the hyperbolic component of period 1, this clearly follows from Lemma 3.4.4 and Corollary 3.4.10.

Let c_0 be the center of the hyperbolic component H of odd period $k \neq 1$. There are $d+1$ fixed points (under $f_{c_0}^k$) on the boundary of the characteristic Fatou component of f_{c_0} , d of which are the landing points of one dynamical ray (of period k) each, while one is the landing point of exactly two dynamical rays (each of period $2k$). By Lemmas 3.4.6, 3.4.7 and 3.4.13, all the corresponding parameter rays must accumulate on ∂H , but not on cusps. Further, two different parameter rays cannot accumulate on the same co-root parabolic arc (Corollary 3.3.7). Therefore, the d parameter rays at angles of period k accumulate on d distinct co-root arcs of H . The two parameter rays at angles of period $2k$ must accumulate on the remaining root arc on ∂H (by Lemma 3.4.4, there are at most $d+1$ arcs). Thus, there are exactly d co-root arcs and one root arc. $\square$

Corollary 3.4.17 (Bijection Between Dynamical and Parameter Rays). *Let H be a hyperbolic component of odd period $k(\neq 1)$ with center c_0 . The characteristic Fatou component U_{c_0} of f_{c_0} has exactly d dynamical co-roots and exactly 1 dynamical root on its boundary. Let $S = \{\vartheta_1, \vartheta_2, \dots, \vartheta_d\}$ (each angle of period k) be the set of angles of the dynamical rays that land at the d dynamical co-roots on ∂U_{c_0} , and let $S' = \{\alpha_1, \alpha_2\}$ (each angle of period $2k$) be the set of angles of the dynamical rays that land at the unique dynamical root on ∂U_{c_0} .*

Then, ∂H has exactly d co-root arcs $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_d$ and exactly 1 root arc $\mathcal{C}_{d+1}$, such that $L_{\mathcal{M}_d^}(\vartheta_i) \subset \mathcal{C}_i$ (for $i \in \{1, 2, \dots, d\}$), and $L_{\mathcal{M}_d^*}(\alpha_1) \cup L_{\mathcal{M}_d^*}(\alpha_2) \subset \mathcal{C}_{d+1}$.*

Corollary 3.4.18. *Let H be a hyperbolic component of odd period k . Let $\mathcal{R}_{\alpha_1}^d, \mathcal{R}_{\alpha_2}^d$ be the two parameter rays at $2k$ -periodic angles that accumulate on the unique root arc, and $\mathcal{R}_{\vartheta_1}^d, \mathcal{R}_{\vartheta_2}^d$ be the two parameter rays at k -periodic angles that accumulate on the two co-root arcs adjacent to the root arc of H such that $\alpha_1 < \vartheta_1 < \vartheta_2 < \alpha_2$. Then, $(1+d^k) \cdot (\vartheta_1 - \alpha_1) = (\alpha_2 - \alpha_1) = (1+d^k) \cdot (\alpha_2 - \vartheta_2)$.*

Proof. Let c_1 and c_2 be the two cusps at the ends of the unique root arc of H and $\mathcal{P}_1, \mathcal{P}_2$ be the parabolic orbit portraits of f_{c_1}, f_{c_2} respectively. The

lengths of the characteristic arcs of $\mathcal{P}_1, \mathcal{P}_2$ are $\vartheta_1 - \alpha_1$ and $\alpha_2 - \vartheta_2$ respectively (possibly after renumbering c_1 and c_2). By Lemma 2.2.6, $(1 + d^k) \cdot (\vartheta_1 - \alpha_1) = (\alpha_2 - \alpha_1) = (1 + d^k) \cdot (\alpha_2 - \vartheta_2)$. $\square$

Remark 3.4.19. *We have shown that exactly 2 rays at k -periodic angles and 2 rays at $2k$ -periodic angles accumulate on the boundary of a hyperbolic component H of odd period $k > 1$ of the tricorn. It is easy to see that starting from one of the rays at a k -periodic angle accumulating on ∂H , one can find the other ray (at a k -periodic angle) by an algorithm similar to that of finding conjugate rays of the Mandelbrot set (see [BS02]). Hence, given any parameter ray accumulating on ∂H , this algorithm together with Corollary 3.4.18 allows us to determine all the other rays accumulating there.*

3.5 Discontinuity of Landing Points

It is known that the landing point of a rational dynamical ray of a unicritical (holomorphic) polynomial behaves nicely with respect to the parameter. More precisely, for any rational angle ϑ , the landing point of the dynamical ray at angle ϑ (of $z^d + c$) depends continuously on the parameter c on the entire subset of the complex plane for which the ray lands (see Proposition [Sch00, Proposition 5.1], Theorem [EMS14, Theorem 4.1]). In this section, we utilize our work on orbit portraits to show that this property fails to hold at each parabolic arc of the multicorns.

Throughout this section, we assume the following:

- H is a hyperbolic component of odd period k of $\mathcal{M}_d^*$.
- $\mathcal{C}_0$ is a root arc on the boundary of H .
- $\mathcal{C}_1$ and $\mathcal{C}_2$ are two co-root arcs on the boundary of H (such that $\mathcal{C}_0$ and $\mathcal{C}_2$ are adjacent to $\mathcal{C}_1$).
- Let α_1 and α_2 be the angles of the two dynamical rays which land at the characteristic parabolic point of f_c for every parameter $c \in \mathcal{C}_0$. Hence, the two parameter rays at angles α_1 and α_2 accumulate on $\mathcal{C}_0$.
- Let ϑ_1 (respectively ϑ_2) be the angle of the unique dynamical ray that lands at the characteristic parabolic point of f_c for every parameter $c \in \mathcal{C}_1$ (respectively $\mathcal{C}_2$). Hence, the parameter ray at angle ϑ_1 (respectively ϑ_2) accumulates on $\mathcal{C}_1$ (respectively $\mathcal{C}_2$).
- $\mathcal{P}_1$ is the parabolic orbit portrait of the cusp point f_{c_1} where $\mathcal{C}_0$ and $\mathcal{C}_1$ meet such that the characteristic angles are $\{\vartheta_1, \alpha_1\}$.
- $\mathcal{P}_2$ is the parabolic orbit portrait of the cusp point f_{c_2} where $\mathcal{C}_1$ and $\mathcal{C}_2$ meet such that the characteristic angles are $\{\vartheta_1, \vartheta_2\}$.

- H' (respectively H'') is the hyperbolic component of period $2k$ that bifurcates from the arcs $\mathcal{C}_0$ and $\mathcal{C}_1$ (respectively $\mathcal{C}_1$ and $\mathcal{C}_2$).
- The $\mathcal{P}_1$ -wake (respectively $\mathcal{P}_2$ -wake) is defined as: $W_{\mathcal{P}_1} :=$ The connected component of $\mathbb{C} \setminus \{\mathcal{R}_{\vartheta_1}^d \cup \mathcal{R}_{\alpha_1}^d \cup \mathcal{C}_0 \cup \mathcal{C}_1 \cup \{c_1\}\}$ not containing 0. (respectively $W_{\mathcal{P}_2} :=$ The connected component of $\mathbb{C} \setminus \{\mathcal{R}_{\vartheta_1}^d \cup \mathcal{R}_{\vartheta_2}^d \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{c_2\}\}$ not containing 0.)
- For any $\vartheta \in \mathbb{Q}/\mathbb{Z}$, define the set $S(\vartheta) = \{c \in \mathbb{C} : R_c(\vartheta) \text{ lands}\}$ and the map $L_\vartheta : S(\vartheta) \rightarrow \mathbb{C}$ as $L_\vartheta(c) =$ The landing point of the dynamical ray $R_c(\vartheta)$.

Lemma 3.5.1. *The set of angles of dynamical rays landing at the dynamical root of the characteristic Fatou component of the center of H' (respectively H'') is given by $\{\vartheta_1, \alpha_1, \alpha_2\}$ (respectively $\{\vartheta_1, \vartheta_2\}$).*

Proof. We prove the result for H' . The other case can be proved analogously.

The dynamical root $\tilde{z}$ of the characteristic Fatou component of the center $\tilde{c}$ of H' has period k , and it lies on the boundary of exactly two Fatou components of period $2k$ each. By [NS03, Corollary 4.2], this dynamical root point $\tilde{z}$ is the landing point either of exactly two rays of period k (which are fixed by the first return map $f_{\tilde{c}}^{\circ k}$ of $\tilde{z}$), or of exactly one ray of period k and exactly two rays of period $2k$ (the first one is fixed; the latter two are interchanged by the first return map $f_{\tilde{c}}^{\circ k}$ of $\tilde{z}$). Let $\mathcal{A}$ be the set of angles of the dynamical rays of $f_{\tilde{c}}$ landing at $\tilde{z}$. Observe that $\tilde{z}$ is a repelling periodic point, and H' does not intersect any parameter ray or any parabolic parameter. Hence, Lemma 3.1.3 implies that there exists a real-analytic function $z : H' \rightarrow \mathbb{C}$ such that $z(\tilde{c}) = \tilde{z}$, $z(c)$ is the dynamical root of the characteristic Fatou component of f_c , and the dynamical rays $\{R_c(t) : t \in \mathcal{A}\}$ land at $z(c)$, for all $c \in H'$.

The characteristic parabolic point of f_{c_1} is a triple fixed point of $f_{c_1}^{\circ 2k}$ which splits into two attracting points (both of period $2k$ under f_{c_1}) and a repelling point (of period k under f_{c_1}) when perturbed into H' . In fact, this repelling point is the dynamical root point of the characteristic Fatou component of the perturbed antiholomorphic polynomial. If the dynamical rays $\{R_{c_1}(\beta) : \beta \in \mathcal{A}\}$ land at some repelling periodic point(s) of f_{c_1} , then for nearby parameters, the dynamical rays at the same angles would continue to land at the real-analytic continuations of those repelling points (by Lemma 3.1.3); which would necessarily be different from the dynamical root point of the characteristic Fatou component: a contradiction. Therefore, the set of rays $\{R_{c_1}(\beta) : \beta \in \mathcal{A}\}$ land at the parabolic cycle of f_{c_1} . It follows from the orbit separation lemma (Lemma 3.1.14) that these rays must land at the characteristic parabolic point of f_{c_1} .

We know by Lemma 3.3.11 that the three dynamical rays $R_{c_1}(\vartheta_1)$, $R_{c_1}(\alpha_1)$ and $R_{c_1}(\alpha_2)$ land at the characteristic parabolic point of f_{c_1} , and no other ray

lands there. Therefore, $\mathcal{A} \subseteq \{\vartheta_1, \alpha_1, \alpha_2\}$. It now follows that $\mathcal{A} = \{\vartheta_1, \alpha_1, \alpha_2\}$. $\square$

The external angle $t(c)$ of every parameter $c \in W_{\mathcal{P}_1} \cap (\mathbb{C} \setminus \mathcal{M}_d^*)$ (respectively $c \in W_{\mathcal{P}_2} \cap (\mathbb{C} \setminus \mathcal{M}_d^*)$) lies in the characteristic arc (α_1, ϑ_1) (respectively $(\vartheta_1, \vartheta_2)$) of $\mathcal{P}_1$ (respectively $\mathcal{P}_2$). It follows from the proof of Theorem 2.2.2 that f_c has a repelling periodic orbit with associated orbit portrait $\mathcal{P}_1$ (respectively $\mathcal{P}_2$) for every $c \in W_{\mathcal{P}_1} \cap (\mathbb{C} \setminus \mathcal{M}_d^*)$ (respectively $c \in W_{\mathcal{P}_2} \cap (\mathbb{C} \setminus \mathcal{M}_d^*)$).

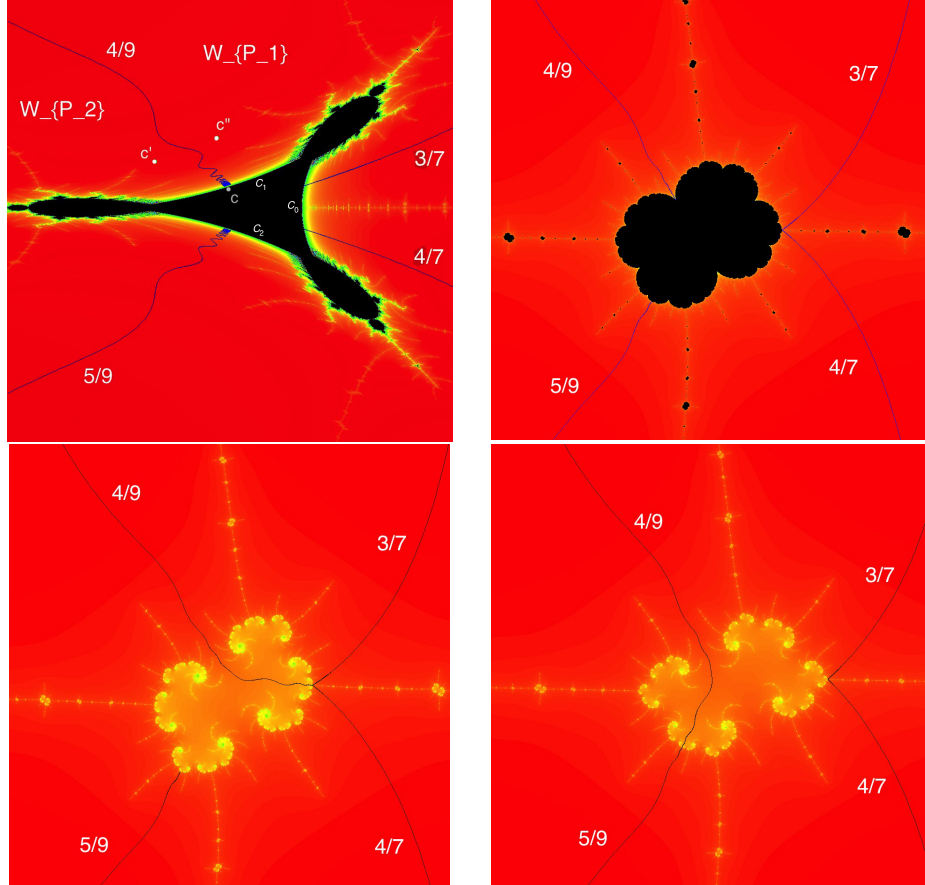


Figure 3.3: Clock-wise from top left: The points c , c' and c'' lie on the parabolic arc $\mathcal{C}_1$ and in the two wakes $W_{\mathcal{P}_2}$ and $W_{\mathcal{P}_1}$ respectively. The dynamical planes of f_c , $f_{c'}$ and $f_{c''}$ exhibit the jump discontinuity of the landing point of the dynamical $4/9$ -ray under perturbations into two different wakes.

The next theorem shows that the landing point of the dynamical ray at angle ϑ_1 undergoes a jump discontinuity as one approaches the parabolic arc $\mathcal{C}_1$ from the wakes $W_{\mathcal{P}_1}$ and $W_{\mathcal{P}_2}$. The idea is simple: there exist parameters arbitrarily close to $\mathcal{C}_1$ for which the dynamical ϑ_1 -ray and one (or two) other fixed dynamical ray(s) land at a common point, depending on whether the

parameter is in the wake $W_{\mathcal{P}_2}$ or $W_{\mathcal{P}_1}$. But on the parabolic arc $\mathcal{C}_1$, which is on the boundary on these wakes, the dynamical ϑ_1 -ray lands alone.

Theorem 3.5.2 (Landing Point Depends Discontinuously on Parameters). *The map $L_{\vartheta_1} : S(\vartheta_1) \mapsto \mathbb{C}$ is discontinuous at every point of $\mathcal{C}_1$.*

Proof. We first note that the rays $R_c(\vartheta_2)$, $R_c(\alpha_1)$ and $R_c(\alpha_2)$ land on repelling periodic points for each $c \in \mathcal{C}_1$. Therefore, each of the maps L_{ϑ_2} , L_{α_1} and L_{α_2} is continuous in a neighborhood of $\mathcal{C}_1$ (this neighborhood does not contain the cusps).

By Theorem 3.2.1 and Lemma 3.2.3, the Ecalle height of the critical value parametrizes the parabolic arc $\mathcal{C}_1$; i.e. there is a real-analytic bijection $c : \mathbb{R} \rightarrow \mathcal{C}_1$ such that $\lim_{t \rightarrow -\infty} c(t) = c_2$ and $\lim_{t \rightarrow \infty} c(t) = c_1$. We assume that $L_{\mathcal{M}_d^*}(\vartheta_1) = c[a, b]$ (with $b \geq a$).

For any $t \in (-\infty, b]$, every neighborhood $c(t)$ contains parameters $c' \in W_{\mathcal{P}_2}$ such that $R_{c'}(\vartheta_1)$ and $R_{c'}(\vartheta_2)$ land together in the dynamical plane of $f_{c'}$; i.e. $L_{\vartheta_1}(c') = L_{\vartheta_2}(c')$ (see Figure 3.3), and L_{ϑ_2} is continuous in this neighborhood. If L_{ϑ_1} was continuous at $c(t)$, this would imply that $L_{\vartheta_1}(c(t)) = L_{\vartheta_2}(c(t))$. This contradicts the fact that only one dynamical ray lands at the characteristic parabolic point of $c(t)$ (since $c(t)$ lies on a co-root arc). Therefore, L_{ϑ_1} is discontinuous on $c(-\infty, b]$.

On the other hand, if $t \in [a, \infty)$, every neighborhood $c(t)$ would contain parameters $c'' \in W_{\mathcal{P}_1}$ such that $R_{c''}(\vartheta_1)$ and $R_{c''}(\alpha_1)$ land together in the dynamical plane of $f_{c''}$; i.e. $L_{\vartheta_1}(c'') = L_{\alpha_1}(c'')$ (see Figure 3.3), and L_{α_1} is continuous in this neighborhood. If L_{ϑ_1} was continuous at $c(t)$, we would have $L_{\vartheta_1}(c(t)) = L_{\alpha_1}(c(t))$. This again contradicts the fact that only one dynamical ray lands at the characteristic parabolic point of $c(t)$ (since $c(t)$ lies on a co-root arc). Therefore, L_{ϑ_1} is discontinuous on $c[a, \infty)$. $\square$

Remark 3.5.3. *It follows from the proof of Theorem 3.5.2 that the function L_{ϑ_1} exhibits a ‘double’ discontinuity on the accumulation set $L_{\mathcal{M}_d^*}(\vartheta_1)$ of the parameter ray $\mathcal{R}_{\vartheta_1}^d$: for any $c \in L_{\mathcal{M}_d^*}(\vartheta_1) \subset \mathcal{C}_1$, any neighborhood of c contains parameters c' and c'' such that the landing points of the dynamical rays $R_{c'}(\vartheta_1)$ and $R_{c''}(\vartheta_1)$ ‘jump’ in two different directions.*

Corollary 3.5.4. *The set of accumulation points of the parameter ray $\mathcal{R}_{\vartheta_1}^d$ is $c[a, b] = \{c \in \mathcal{C}_1 : \text{every neighborhood of } c \text{ contains } c' \text{ and } c'' \text{ such that } L_{\vartheta_1}(c') = L_{\vartheta_2}(c') \text{ and } L_{\vartheta_1}(c'') = L_{\alpha_1}(c'')\}$.*

3.6 Number of Hyperbolic Components

In the final section of this chapter, we will prove Theorem 1.2.3, which gives a formula for the number of hyperbolic components of period k of $\mathcal{M}_d^*$. This is done by counting the number of parameter rays at k -periodic angles (under

multiplication by $-d$) that land on the boundary of a hyperbolic component of the same period.

For any $k \in \mathbb{N}, k > 2$, let $\varphi(d, k)$ denote the number of angles in $\mathbb{Q}/\mathbb{Z}$ of period k under multiplication by $-d$. It is easy to check that this is equal to the number of angles in $\mathbb{Q}/\mathbb{Z}$ of period k under multiplication by d . It is well-known that the number of hyperbolic components of period k in $\mathcal{M}_d$ is given by $s_{d,k} = \varphi(d, k)/d$ (see [EMS14]).

Lemma 3.6.1. *For any odd $k (\neq 1)$, there are exactly $\varphi(d, k)/d$ hyperbolic components of period k in $\mathcal{M}_d^*$. Thus, $s_{d,k}^* = s_{d,k}$.*

Proof. We have seen in Section 3.4 that every parameter ray at a k -periodic angle (k odd, $k \neq 1$) lands/accumulates on a sub-arc of a parabolic arc of some hyperbolic component of period k . Further, every hyperbolic component of odd period k absorbs exactly d parameter rays at k -periodic angles. A single ray can not accumulate on the boundary of two distinct hyperbolic components of period k (the accumulation set of a ray is connected). Therefore, every hyperbolic component of period k has d rays of its own, and this accounts for all the parameter rays at k -periodic angles. Hence, $s_{d,k}^* = \varphi(d, k)/d$. The second statement follows from the proof of [EMS14, Corollary 5.4]. $\square$

Now we turn our attention to the hyperbolic components of even periods. The first step is to discuss the number of rays landing on the boundary of an even-periodic hyperbolic component that bifurcates from an odd periodic one. We stick to the terminologies of Section 3.5.

Lemma 3.6.2. *1) Let k be an odd integer. Every rational parameter ray at a $2k$ -periodic angle ϑ either lands at a parabolic parameter on the boundary of a hyperbolic component of period $2k$ or lands/accumulates on a sub-arc of a parabolic root arc of a hyperbolic component of period k .*

2) Let ϑ be periodic with period $4k$, for some $k \in \mathbb{N}$. Then the parameter ray $\mathcal{R}_\vartheta^d$ lands at a parabolic parameter of even period. In particular, the landing point lies on the boundary of a hyperbolic component of period $4k$.

Proof. 1) Let ϑ be of period $2k$ under multiplication by $-d$. If c is an accumulation point of the parameter ray $\mathcal{R}_\vartheta^d$, then $R_c(\vartheta)$ lands on a parabolic point of period r in the dynamical plane of f_c such that $r|2k$ (by Lemma 3.3.5). By Lemma 3.1.7, there are only finitely many such c with even values of r . If r is odd, Lemma 2.1.7 says that $r = k$. Therefore, the accumulation set of $\mathcal{R}_\vartheta^d$ is contained in the union of the (finitely many) parabolic arcs of period k and a finite number of parabolic parameters of even period and ray period $2k$. Since the set of accumulation points of a ray is connected, we conclude that $\mathcal{R}_\vartheta^d$ either lands/accumulates on a sub-arc of a single parabolic (root) arc of a hyperbolic component of period k or lands at a parabolic parameter of even period r with ray period $2k$. In the latter case, if $r = 2k$, it follows from

Theorem 3.1.1 that the landing point lies on the boundary of some hyperbolic component of period $2k$. On the other hand if r is a proper divisor of $2k$, then it is not hard to show that the multiplier of this r -periodic parabolic cycle is a $2k/r$ -th root of unity. It then follows from Theorem 1.2.1 that the landing point lies on the boundary of a hyperbolic component of period $2k$.

2) Every accumulation point of $\mathcal{R}_\vartheta^d$ is a parabolic parameter of period r with ray period $4k$ such that $r|4k$. By Lemma 2.1.7, r must be even. Since there are only finitely such parabolic parameters (by Lemma 3.1.7), the ray must land at a parabolic parameter of ray period $4k$. The fact that the landing point lies on the boundary of a hyperbolic component of period $4k$ is proved as in the previous case. $\square$

Lemma 3.6.3. *Let H' be a hyperbolic component of period $2k$ bifurcating from a hyperbolic component H of odd period k of $\mathcal{M}_d^*$. Then exactly $d-2$ parameter rays at $2k$ -periodic angles land on the boundary of H' .*

Proof. If a parameter ray $\mathcal{R}_\vartheta^d$ at a $2k$ -periodic angle lands on the boundary of a hyperbolic component H' of period $2k$, then the landing point must be a co-root or root point of H' (compare [EMS14]). It follows that ϑ is contained in the set of angles of the dynamical rays that land at various dynamical root and co-root points on the boundary of the characteristic Fatou component of the center c of H' .

The characteristic Fatou component of f_c has exactly $d-2$ dynamical co-root points and exactly one dynamical root point. This dynamical root point is the landing point of exactly two dynamical ray at $2k$ -periodic angles and a ray at k -periodic angle, by Lemma 3.5.1. But the parameter rays at these angles land on the boundary of the odd periodic hyperbolic component H . On the other hand, each of the dynamical co-root points is the landing point of exactly one dynamical ray at a $2k$ -periodic angle. Therefore, we can assume that $R_c(\vartheta)$ lands at a dynamical co-root of f_c . By the previous lemma, the $d-2$ parameter rays at these angles either land at parabolic parameters on the boundary of a hyperbolic components of period $2k$ or land/accumulate on parabolic root arcs of hyperbolic components of period k .

We claim that they land on hyperbolic components of period $2k$. If not, then such a ray at $2k$ -periodic angle ϑ would accumulate on the root arc of a hyperbolic component H_1 of period k . Then there is hyperbolic component H'_1 of period $2k$ (with center c_1) bifurcating from H_1 such that $R_{c_1}(\vartheta)$ lands at the dynamical root point of the characteristic Fatou component of f_{c_1} , and is a (left) supporting ray thereof. This implies that the post-critically finite holomorphic polynomials f_c^2 and $f_{c_1}^2$ have the same critical portrait. It follows from 3.4.2 that $c = c_1$ and hence, $H' = H'_1$. This is a contradiction since a dynamical ray cannot simultaneously land at a dynamical co-root and a dynamical root point of a Fatou component!

Therefore, all the $d-2$ parameter rays under consideration land on the

boundary of hyperbolic components of period $2k$. But it is easy to check that if one of these rays land on the boundary of some other hyperbolic component of period $2k$, then the center of that component would have the same critical portrait as c . By 3.4.2, this is impossible. Hence, all these $d - 2$ parameter land on the boundary of H' and these are all. $\square$

Lemma 3.6.4. *Let H be a hyperbolic component of even period k that does not bifurcate from an odd period hyperbolic component. Then exactly d parameter rays at k -periodic angles land on the boundary of H .*

Proof. As in the previous lemma, if a parameter ray at an k -periodic angle ϑ lands on the boundary of H , then ϑ is contained in the set of angles of the dynamical rays that land at various dynamical root and co-root points on the boundary of the characteristic Fatou component of the center c of H . The characteristic Fatou component of the center of H has exactly $d - 2$ dynamical co-root points and exactly one dynamical root point. Each of these co-roots is the landing point of a unique periodic dynamical ray of period k . One can argue as in Lemma 3.6.3 that the $d - 2$ parameter rays at these angles land at parabolic parameters on the boundary of H .

The dynamical root point of the characteristic Fatou component of c has a non-trivial orbit portrait. Let its characteristic angles be $\{t^-, t^+\}$. Arguing as in Lemma 3.4.7, we see that the two parameter rays $\mathcal{R}_{t^-}^d$ and $\mathcal{R}_{t^+}^d$ land at a common parabolic parameter on the boundary of a hyperbolic component of period k . Another straight-forward application of 3.4.2 shows that this component must be H . The landing point of $\mathcal{R}_{t^-}^d$ and $\mathcal{R}_{t^+}^d$ is called the root point of H , which is the unique parabolic parameter on ∂H with ray period k with the property that the parabolic cycle disconnects the Julia set. The proof of the fact that $\mathcal{R}_{t^-}^d$ and $\mathcal{R}_{t^+}^d$ are the only parameter rays landing at this root point is analogous to the holomorphic case (see [EMS14], [Sch00]). This completes the proof that exactly d parameter rays at k -periodic angles land on the boundary of H . $\square$

Proof of Theorem 1.2.3. For any odd integer k , the result follows from Lemma 3.6.1.

If k is a multiple of 4, a hyperbolic component of period k can never bifurcate from a hyperbolic component of odd period. In this case, the result follows from Lemma 3.6.4.

In the remaining case when k is twice an odd integer, we have to work a little more. There are exactly $s_{d,k/2}^*$ hyperbolic components of period $k/2$ and each of them absorbs two parameter rays at k -periodic angles. Each of these hyperbolic components of period $k/2$ has $d + 1$ hyperbolic components of period k bifurcating from it. Each of these hyperbolic components of period k absorbs exactly $d - 2$ parameter rays at k -periodic angles. This accounts for $(d + 1)s_{d,k/2}^*$ hyperbolic components of period k and $\{(d - 2)(d + 1) +$

$2\} s_{d,k/2}^* = (d^2 - d) s_{d,k/2}^*$ parameter rays at k -periodic angles. We are still left with $\varphi(d, k) - (d^2 - d) s_{d,k/2}^*$ rays at k -periodic angles and these must land in groups of d rays on the boundaries of hyperbolic components of period k that do not bifurcate from odd period hyperbolic components (by Lemma 3.6.4). This accounts for $\{\varphi(d, k) - (d^2 - d) s_{d,k/2}^*\}/d$ other components of period k and these are all. Combining these, we deduce that

$$\begin{aligned} s_{d,k}^* &= (d+1)s_{d,k/2}^* + \{\varphi(d, k) - (d^2 - d)s_{d,k/2}^*\}/d \\ &= (d+1)s_{d,k/2} + s_{d,k} - (d-1)s_{d,k/2} \\ &= s_{d,k} + 2s_{d,k/2} . \end{aligned}$$

□

Chapter 4

Complexification of Parabolic Arcs

It has transpired through our work in the previous chapter that the existence of abundant parabolic parameters on the boundaries of odd period hyperbolic components of the multicorns is at the heart of many combinatorial and topological differences between the multicorns and the multibrot sets. In this chapter, we will embed each parabolic arc of $\mathcal{M}_d^*$ in a one complex-dimensional family of quasi-conformally conjugate holomorphic polynomials. This result can be thought of as a form of structural stability of the odd period non-degenerate parabolic parameters along an algebraic curve transverse to the multicorns. This allows us to prove Theorem 1.2.4 and Theorem 1.2.5.

These results are available in the arXiv preprint [Muk14].

4.1 An Analytic Family of Q.C. Deformations

Throughout this section, we fix a parabolic arc $\mathcal{C}$ such that for each $c \in \mathcal{C}$, f_c has a k -periodic parabolic cycle. Recall that there is a dynamically defined parametrization of $\mathcal{C}$, which is its critical Ecalle height parametrization $c : \mathbb{R} \rightarrow \mathcal{C}$. We will show that the polynomials on $\mathcal{C}$ can be quasi-conformally deformed to yield an analytic family of q.c. conjugate maps; in particular, they will be structurally stable on a suitable algebraic curve.

We embed our family $f_c(z) = \bar{z}^d + c$, $c \in \mathbb{C}$ in the family of holomorphic polynomials $\mathcal{F}_d = \{P_{a,b}(z) = (z^d + a)^d + b, a, b \in \mathbb{C}\}$. Since $f_c^{\circ 2} = P_{\bar{c},c}$, the connectedness locus $\mathcal{C}(P_{a,b})$ of this family intersects the slice $\{a = \bar{b}\}$ in $\mathcal{M}_d^*$.

It will be useful to have the following characterization of the elements of $\mathcal{F}_d$ among all monic centered polynomials of degree d^2 .

Lemma 4.1.1. *Let f be a monic centered polynomial of degree d^2 . Then the following are equivalent:*

1. $f \in \mathcal{F}_d$ with $a \neq 0$.

2. f has exactly $d+1$ distinct critical points $\{\alpha_1, \alpha_2, \dots, \alpha_{d+1}\}$ with $\deg_{\alpha_i}(f) = d$ for each i and such that $f(\alpha_1) = f(\alpha_2) = \dots = f(\alpha_d)$.

Proof. (1) $\implies$ (2). The critical points of any $f \in \mathcal{F}_d$ are 0 and the d roots of the equation $z^d + a = 0$. Since $a \neq 0$, these points are all distinct and f has local degree d at each of them. The other property is immediate.

(2) $\implies$ (1). Let $b = f(\alpha_1) = f(\alpha_2) = \dots = f(\alpha_d)$. Since each α_i ($i = 1, 2, \dots, d$) maps in a d -to-1 fashion to b and the degree of f is d^2 , these must be all the pre-images of b . Therefore, $f(z) = p(z)^d + b$, where $p(z) = (z - \alpha_1) \cdots (z - \alpha_d)$ (here we've used the fact that f is monic). Note that the critical points of f are precisely the zeroes and the critical points of p . Since we have used up the d distinct zeroes of p and are left with only one critical point of f , it follows that this critical point α_{d+1} must be the only critical point of p . Hence, p must be unicritical. A brief computation (using the fact that f is centered) now shows that $p(z) = z^d + a$, for some $a \in \mathbb{C}$. The fact that all the α_i 's are distinct tells that a must be non-zero. Thus, we have shown that $f(z) = (z^d + a)^d + b$ with $a \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$. $\square$

Remark 4.1.2. Condition (2) of the previous lemma is preserved under topological conjugacies.

Since every antiholomorphic polynomial on the parabolic arc has a parabolic cycle of exact period k and multiplier 1, the parabolic arc $\mathcal{C}$ is contained in the algebraic curve:

$$P_k = \{(a, b) \in \mathbb{C}^2 : \Psi(a, b) := \text{disc}_z \left(P_{a,b}^{\circ k}(z) - z \right) = 0\}$$

where $\text{disc}_z(p)$ of a polynomial p in one complex variable z denotes the discriminant of p (see [DF03, §14.6]).

Since an affine algebraic curve has at most finitely many singular points, all but finitely many points of the parabolic arc are non-singular points of this algebraic set; i.e. the algebraic set is locally a graph near such points. The parabolic arc $\mathcal{C}$ is naturally embedded in $\mathbb{C}^2$ by the map $z \mapsto (\bar{z}, z)$.

We are now in a position to prove the main technical lemma, which allows us to embed the parabolic arc $\mathcal{C}$ in a complex one-dimensional deformation space contained in P_k . The idea of the proof is as follows. For any $c_0 \in \mathcal{C}$, the second iterate $P_{\bar{c}_0, c_0} = f_{c_0}^{\circ 2}$ of the antiholomorphic polynomial f_{c_0} is a holomorphic polynomial with two infinite critical orbits. Both these critical orbits are attracted by the unique parabolic cycle of $P_{\bar{c}_0, c_0}$, and hence we can choose two representatives of these two critical orbits in a fundamental domain of an attracting petal of $P_{\bar{c}_0, c_0}$. The difference of the attracting Fatou coordinates of these two chosen representatives (the difference does not depend on the choice of the representatives as long as they are chosen from the same fundamental domain) turn out to be a conformal conjugacy invariant of the polynomial. This invariant will be called the *Fatou vector* of the polynomial.

We will deform the polynomial in such a way that the qualitative behavior of the dynamics remain the same, but the Fatou vector varies over a bi-infinite strip. Thus, each deformation would give us a topologically conjugate, but conformally different polynomial.

Lemma 4.1.3 (Extending the deformation). *Let $c_0 = c(0) \in \mathcal{C}$. There exists an injective holomorphic map*

$$F : \{w = u + iv \in \mathbb{C} : |v| < \frac{1}{4}\} \rightarrow P_k, \\ w \mapsto (a(w), b(w))$$

with $F(0) = (\overline{c_0}, c_0)$ such that for any $w \in \mathbb{R}$, $F(w) = (\overline{c(w)}, c(w))$ where $c : \mathbb{R} \rightarrow \mathcal{C}$ is the critical Ecalle height parametrization of the parabolic arc. Further, all the polynomials $P_{a(w), b(w)}$ have q.c.-conjugate (but not conformally conjugate) dynamics.

Proof. We construct a larger class of deformations which strictly contains the deformations constructed in Theorem 3.2.1. Choose the attracting Fatou coordinate $\Phi_0 : z \mapsto \zeta$ (at the parabolic point on the boundary of the unique Fatou component of f_{c_0} that contains c_0) such that the unique geodesic invariant under the antiholomorphic dynamics maps to the real line and the critical value c_0 has Ecalle coordinates $\frac{1}{4}$ (this is possible since $c_0 = c(0)$). The polynomial $f_{c_0}^{\circ 2} = P_{\overline{c_0}, c_0}$ has two critical values c_0 and $\overline{c_0}^d + c_0$. The map $f_{c_0}^{\circ k-1} = P_{\overline{c_0}, c_0}^{\circ \frac{k-1}{2}}$ sends $\overline{c_0}^d + c_0$ to the Fatou component containing c_0 . The Ecalle coordinates of $P_{\overline{c_0}, c_0}^{\circ \frac{k-1}{2}}(\overline{c_0}^d + c_0) = f_{c_0}^{\circ k}(c_0)$ is $\frac{3}{4}$. We will simultaneously change the Ecalle coordinates of c_0 and $f_{c_0}^{\circ k}(c_0)$ in a controlled way, so that each perturbation gives a conformally different polynomial.

Setting $\zeta = x + iy$, we can change the conformal structure within the Ecalle cylinder by the quasi-conformal homeomorphism of $[0, 1] \times \mathbb{R}$ (compare Figure 4.1):

$$L_w : \zeta \mapsto \begin{cases} \zeta + 4iw x & \text{if } 0 \leq x \leq 1/4 \\ \zeta + 2iw(1 - 2x) & \text{if } 1/4 \leq x \leq 1/2 \\ \zeta - 2iw(2x - 1) & \text{if } 1/2 \leq x \leq 3/4 \\ \zeta - 4iw(1 - x) & \text{if } 3/4 \leq x \leq 1 \end{cases}$$

for $S = \{w = u + iv \in \mathbb{C} : |v| < \frac{1}{4}\}$

Translating the map by positive integers, we get a q.c. homeomorphism of a right-half plane that commutes with the translation $z \mapsto z + 1$. By the coordinate change $z \mapsto \zeta$, we can transport this Beltrami form (defined by this quasi-conformal homeomorphism) into all the attracting petals, and it is forward invariant under $f_{c_0}^{\circ 2} = P_{\overline{c_0}, c_0}$. It is easy to make it backward invariant by pulling it back along the dynamics. Extending it by the zero Beltrami form outside of the entire parabolic basin, we obtain an $P_{\overline{c_0}, c_0}$ -invariant Beltrami form. Using the Measurable Riemann Mapping Theorem with parameters, we

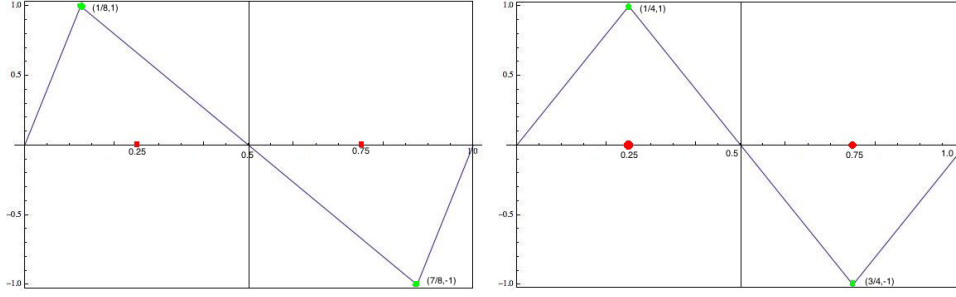


Figure 4.1: Pictorial representation of the image of $[0, 1]$ under the quasi-conformal map L_w ; for $w = 1 + i/8$ (left) and $w = 1$ (right). The Fatou coordinates of c_0 and $f_{c_0}^{\circ k}(c_0)$ are $1/4$ and $3/4$ respectively. For $w = 1 + i/8$, $L_w(1/4) = 1/8 + i$ and $L_w(3/4) = 7/8 - i$, and for $w = 1$, $L_w(1/4) = 1/4 + i$ and $L_w(3/4) = 3/4 - i$. Observe that L_w commutes with $z \mapsto \bar{z} + 1/2$ only when $w \in \mathbb{R}$.

obtain a qc-map φ_w integrating this Beltrami form. Furthermore, if we normalize φ_w such that the conjugated map $\varphi_w \circ P_{\bar{c}_0, c_0} \circ \varphi_w^{-1}$ is always a monic and centered polynomial, then the coefficients of this newly obtained polynomial will depend holomorphically on w (since the Beltrami form depends complex analytically on w).

We need to check that this new polynomial belongs to our family $\mathcal{F}_d$. But this readily follows from Lemma 4.1.1 and the remark thereafter. Therefore, this new polynomial must be of the form $P_{a(w), b(w)} = (z^d + a(w))^d + b(w)$. Since the coefficients of $P_{a(w), b(w)}$ depend holomorphically on w , the maps $w \mapsto a(w)$ and $w \mapsto b(w)$ are holomorphic. Its Fatou coordinate is given by $\Phi_w = L_w \circ \Phi_0 \circ \varphi_w^{-1}$. Note that the Ecalle coordinates of the two representatives of the two critical orbits of $\varphi_w \circ P_{\bar{c}_0, c_0} \circ \varphi_w^{-1}$ are $L_w(\frac{1}{4}) = \frac{1}{4} + iw$ and $L_w(\frac{3}{4}) = \frac{3}{4} - iw$.

Thus, we obtain a complex analytic map $F : S \rightarrow \mathbb{C}^2, w \mapsto (a(w), b(w))$. For real values of w , the map L_w commutes with the map $z \mapsto \bar{z} + 1/2$, and hence, the corresponding Beltrami form is invariant under the antiholomorphic polynomial f_{c_0} (recall that Φ_0 conjugates $f_{c_0}^{\circ k}$ to $z \mapsto \bar{z} + 1/2$ in the attracting petal). Therefore, $\varphi_w \circ f_{c_0} \circ \varphi_w^{-1}$ is again a unicritical antiholomorphic polynomial in the family $\{f_c\}_{c \in \mathbb{C}}$. In particular, for $w \in \mathbb{R}$, the critical Ecalle height of $\varphi_w \circ f_{c_0} \circ \varphi_w^{-1}$ is w . Therefore, $F(w) = (\overline{c(w)}, c(w)) \forall w \in \mathbb{R}$.

By construction, all the $P_{a(w), b(w)}$'s are quasi-conformally conjugate to $P_{\bar{c}_0, c_0}$, and hence to each other. We now show that they are all conformally distinct. Define the *Fatou vector* of $P_{a(w), b(w)}$ to be the quantity $\Phi_w(P_{a(w), b(w)}^{\circ \frac{k-1}{2}}(a(w)^d + b(w))) - \Phi_w(b(w))$, where Φ_w is a Fatou coordinate of $P_{a(w), b(w)}$. Since the Fatou coordinate is unique up to addition by a complex constant, the Fatou vector defined above is a conformal conjugacy invariant. The Fatou vector of $P_{a(w), b(w)}$ is given by $\frac{1}{2} - 2iw$, which is different for different values of w . Hence,

the polynomials $P_{a(w),b(w)}$ are conformally nonequivalent. In particular, the map F is injective.

It remains to show that for any $w \in S$, $(a(w), b(w)) \in P_k$. It is easy to see that each $P_{a(w),b(w)}$ has a parabolic cycle of exact period k (both critical orbits are contained in the Fatou set and converge to a k -periodic orbit sitting on the Julia set). Also, the first return map $P_{a(w),b(w)}^{\circ k}$ leaves the unique petal attached to every parabolic point invariant; so the multiplier of the parabolic cycle must be 1. Hence, $(a(w), b(w)) \in P_k$. $\square$

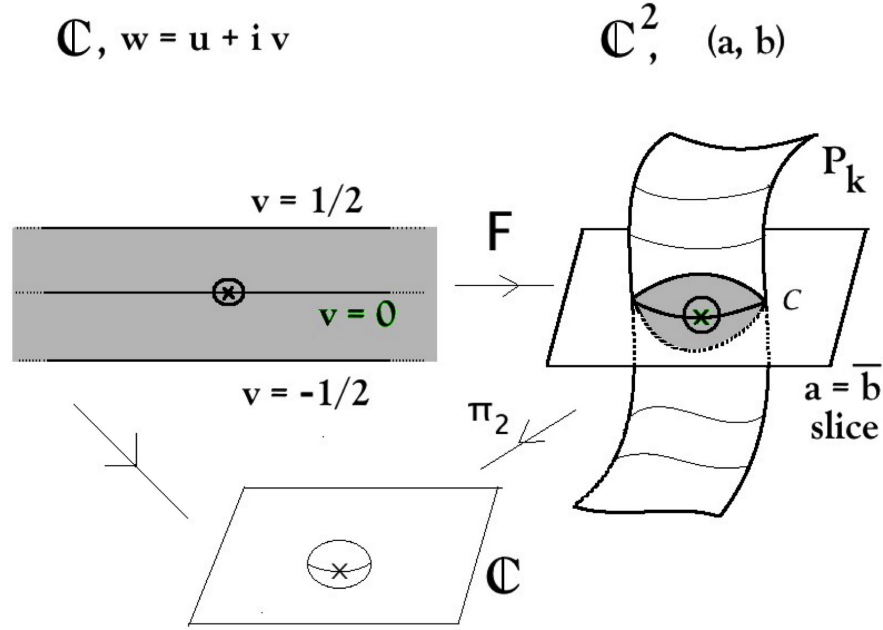


Figure 4.2: $\pi_2 \circ F : w \mapsto b(w)$ is injective in a neighborhood of $\tilde{u}$ for all but possibly finitely many $\tilde{u} \in \mathbb{R}$.

Remark 4.1.4. *It was possible to construct this larger class of deformation since we worked with the polynomial $f_{c_0}^{\circ 2}$ viewing it as a member of the family $\mathcal{F}_d = \{P_{a,b}(z) = (z^d + a)^d + b, a, b \in \mathbb{C}\}$. Working in the family $\{f_c\}_{c \in \mathbb{C}}$ of unicritical antiholomorphic polynomials would have only allowed us to construct the real one-dimensional parabolic arcs, as was done in Theorem 3.2.1. Indeed, the Beltrami form constructed in the previous lemma is invariant under $f_{c_0}^{\circ 2}$ for all $w \in S = \{w = u + iv \in \mathbb{C} : |v| < \frac{1}{4}\}$, but it is invariant under f_{c_0} only when w is real. This is because we obtained the quasi-conformal deformations via the map L_w , which commutes with $z \mapsto \bar{z} + 1/2$ only when $w \in \mathbb{R}$.*

Proof of Theorem 1.2.5. Part (i) follows from Lemma 4.1.3. The required map φ is given by b .

We will show that if $(\overline{c(\tilde{u})}, c(\tilde{u}))$ is a non-singular point of the algebraic curve P_k , then the map $\mathbb{R} \ni h \mapsto c(h)$ has a non-vanishing derivative at $\tilde{u}$. Since any affine algebraic curve has at most finitely many singular points, this will complete the proof of the theorem.

By the definition of non-singularity, one of the partial derivatives $\frac{\partial \Psi}{\partial a}$ or $\frac{\partial \Psi}{\partial b}$ is non-zero at $(\overline{c(\tilde{u})}, c(\tilde{u}))$. Let $\frac{\partial \Psi}{\partial a}(\overline{c(\tilde{u})}, c(\tilde{u})) \neq 0$. Then there exists $\varepsilon_1, \varepsilon_2 > 0$ and a holomorphic map $g : B(c(\tilde{u}), \varepsilon_1) \rightarrow B(\overline{c(\tilde{u})}, \varepsilon_2)$ such that $g(c(\tilde{u})) = \overline{c(\tilde{u})}$ and $\Psi(a, b) = 0$ for some $(a, b) \in B(\overline{c(\tilde{u})}, \varepsilon_1) \times B(c(\tilde{u}), \varepsilon_2)$ if and only if $a = g(b)$. Therefore, the projection map $\pi_2 : (a, b) \mapsto b$ is an injective holomorphic map on an open neighborhood $U \subset P_k$ (in the subspace topology) of $(\overline{c(\tilde{u})}, c(\tilde{u}))$. It follows from Lemma 4.1.3 that the map $\pi_2 \circ F : w \mapsto b(w)$ is an injective holomorphic map on an open neighborhood $B(\tilde{u}, \varepsilon_3) \subset \mathbb{C}$. Hence, it has a non-vanishing derivative at $\tilde{u}$, i.e. $b'(\tilde{u}) \neq 0$.

Writing $b(w) = b(u+iv) = b_1(u+iv) + ib_2(u+iv)$ and $c(u) = c_1(u) + ic_2(u)$, we see that

$$\begin{aligned} b'(\tilde{u}) &= \frac{\partial b_1}{\partial u}(\tilde{u}) + i \frac{\partial b_2}{\partial u}(\tilde{u}) \\ &= (c_1)'(\tilde{u}) + i(c_2)'(\tilde{u}) . \quad [\text{For real } u, b(u) = c(u)] \end{aligned}$$

It follows that $((c_1)'(\tilde{u}), (c_2)'(\tilde{u})) \neq (0, 0)$, i.e. $c'(\tilde{u}) \neq 0$. $\square$

Remark 4.1.5. *One can easily compute that the algebraic curve P_1 of the family $\mathcal{F}_2$ is non-singular at each point of the parabolic arcs of period 1 of the tricorn. Hence, the critical Ecalle height parametrization is indeed non-singular for the period 1 parabolic arcs of the tricorn. However, we conjecture that the critical Ecalle height parametrization of any parabolic arc of $\mathcal{M}_d^*$ is non-singular everywhere. This would follow if we could prove that the algebraic curves P_k considered above are non-singular at every point of a parabolic arc. We will return to this question in Appendix D.*

4.1.1 $\text{Per}_n(1)$ of Biquadratic Polynomials

We observed that the parabolic arcs of period k of $\mathcal{M}_d^*$ are naturally embedded in the algebraic curve P_k . Moreover, for any c on a parabolic arc of period k , the polynomial $P_{\tilde{c}, c}$ is structurally stable along the curve P_k .

The curve P_k is customarily referred to as $\text{Per}_k(1)$ [Mil92] (since the corresponding maps have a k -periodic orbit of multiplier 1). The topology of these curves play an important role in the understanding of the parameter spaces of the maps under consideration. For $k = 1$ and $d = 2$, the curve $\text{Per}_1(1)$ of the family of polynomials $\mathcal{F}_2 = \{P_{a,b}(z) = (z^2 + a)^2 + b, a, b \in \mathbb{C}\}$ has a very simple description. One easily computes that

$$\begin{aligned}\text{Per}_1(1) &= \{(a, b) \in \mathbb{C}^2 : \text{disc}_z(P_{a,b}(z) - z) = 0\} \\ &= \{(a, b) \in \mathbb{C}^2 : h_1(a, b) := 256a^3 + 288ab + 256a^2b^2 + 256b^3 - 27 = 0\}\end{aligned}$$

The set of singular points of $\text{Per}_1(1)$ is $\{(-\frac{3}{4}, -\frac{3}{4}), (-\frac{3}{4}\omega, -\frac{3}{4}\omega^2), (-\frac{3}{4}\omega^2, -\frac{3}{4}\omega)\}$, where ω is a primitive third root of unity. Note that these points correspond precisely to the cusps at the ends of the parabolic arcs of period 1 of the tricorn. In fact, a simple calculation shows that

$$\begin{aligned}\frac{\partial h_1}{\partial a}\big|_{(-\frac{3}{4}, -\frac{3}{4})} &= 0, \quad \frac{\partial h_1}{\partial b}\big|_{(-\frac{3}{4}, -\frac{3}{4})} = 0, \\ \frac{\partial^2 h_1}{\partial a^2}\big|_{(-\frac{3}{4}, -\frac{3}{4})} \frac{\partial^2 h_1}{\partial b^2}\big|_{(-\frac{3}{4}, -\frac{3}{4})} - \left(\frac{\partial^2 h_1}{\partial a \partial b}\big|_{(-\frac{3}{4}, -\frac{3}{4})}\right)^2 &= 0.\end{aligned}$$

The same is true for the other singular points. In particular, each singular point of $\text{Per}_1(1)$ is an ordinary cusp point (i.e. a cusp of the form $x^2 - y^3$ at $(0, 0)$). By the classical degree-genus formula for singular curves, it follows that the genus of $\text{Per}_1(1)$ is 0. Hence, after desingularization (and compactification), $\text{Per}_1(1)$ of the family $\mathcal{F}_2$ is the Riemann sphere $\hat{\mathbb{C}}$.

It will be useful to consider the dynamically defined deformations $\text{Per}_1(r)$, the curve consisting of parameters with a fixed point of multiplier r . More precisely, we define

$$\begin{aligned}\text{Per}_1(r) &= \{(a, b) \in \mathbb{C}^2 : \text{Res}_z(P_{a,b}(z) - z, (P_{a,b})'(z) - r) = 0\} \\ &= \{(a, b) \in \mathbb{C}^2 : h_r(a, b) := 256a^3 + 256ab + 256a^2b^2 + 256b^3 \\ &\quad + 32ab(2r - r^2) + (r^4 - 12r^3 + 48r^2 - 64r) = 0\}\end{aligned}$$

This provides a morsification of the degenerate critical points of h_1 . Let us discuss the situation for h_r (with $r \in (1 - \varepsilon, 1)$, $\varepsilon > 0$ being sufficiently small) near $(-\frac{3}{4}, -\frac{3}{4})$. Each h_r has two non-degenerate critical points close to $(-\frac{3}{4}, -\frac{3}{4})$ (recall that $(-\frac{3}{4}, -\frac{3}{4})$ is a degenerate singularity of h_1 with Milnor number 2; hence for r close to 1, the sum of the Milnor numbers of h_r at points in a small neighborhood of $(-\frac{3}{4}, -\frac{3}{4})$ is 2). One of these two critical points of h_r , say α_r , lies on $\text{Per}_1(r)$. In fact, α_r belongs to a period 1 hyperbolic component (bifurcating from the principal hyperbolic component at $(-\frac{3}{4}, -\frac{3}{4})$) of $\mathcal{F}_2$. Therefore, $h_r(\alpha_r) = 0$, and its Milnor number $\mu(h_r, \alpha_r) = 1$.

Using these information on the family $\text{Per}_1(r)$, we will now show that the cusp points of period n (recall Definition 1.1.16), lying on the boundaries of hyperbolic components of odd period n of the tricorn, are singular points of $\text{Per}_n(1)$ of the family $\mathcal{F}_2$. We will use the concept of the Milnor number of a complex function germ (Compare [Mil68, §7], [Wal04, §6]).

Proposition 4.1.6. *Let $\mathcal{C}$ be a parabolic arc of odd period n of $\mathcal{M}_2^*$, and c_0 be a double parabolic point (cusp point) at the end of $\mathcal{C}$. Then $(\overline{c_0}, c_0)$ is a singular point of $\text{Per}_n(1)$ of $\mathcal{F}_2$.*

Proof. Recall that the set of singular points of $\text{Per}_1(1)$ is $\{(-\frac{3}{4}, -\frac{3}{4}), (-\frac{3}{4}\omega, -\frac{3}{4}\omega^2), (-\frac{3}{4}\omega^2, -\frac{3}{4}\omega)\}$ (where ω is a primitive third root of unity), and these are precisely the cusp points of period 1. We will denote the principal hyperbolic component of the family $\mathcal{F}_2$ by H , and the hyperbolic component (of $\mathcal{F}_2$) of period 1 that bifurcates from H at the parameter $(-\frac{3}{4}, -\frac{3}{4})$ by H_1 .

Let $\tilde{H}$ be the hyperbolic component of period n of $\mathcal{F}_2$ with $\mathcal{C} \subset \partial\tilde{H}$, and $\tilde{H}_1$ be the hyperbolic component of period n of $\mathcal{F}_2$ that bifurcates from $\tilde{H}$ at $(\bar{c}_0, c_0)$. We again consider the 1-parameter family of deformations $\text{Per}_n(r)$ (the curve of parameters with an n -periodic cycle of multiplier r) of $\text{Per}_n(1)$. More precisely, we define $\text{Per}_n(r) := \{(a, b) \in \mathbb{C}^2 : \tilde{h}_r(a, b) = 0\}$, where $\tilde{h}_r(a, b)$ is the square-free part of $\text{Res}_z(P_{a,b}^{\circ n}(z) - z, (P_{a,b}^{\circ n})'(z) - r)$.

By [IK12, Theorem C], the straightening map induces biholomorphisms $\chi : \tilde{H} \rightarrow H$ and $\chi : \tilde{H}_1 \rightarrow H_1$, and $\chi(\bar{c}_0, c_0) = (-\frac{3}{4}, -\frac{3}{4})$ (this is no loss of generality as one of the maps χ , $\omega\chi$ or $\omega^2\chi$ satisfies this property). Since $r < 1$, χ sends a polynomial with an attracting n -cycle of multiplier r to a polynomial with an attracting fixed point of multiplier r ; i.e. $\chi(\tilde{H}_1 \cap \text{Per}_n(r)) = H_1 \cap \text{Per}_1(r)$. Therefore,

$$\begin{aligned} \tilde{H}_1 \cap \text{Per}_n(r) &= \chi^{-1}(H_1 \cap \text{Per}_1(r)) \\ &= \chi^{-1}(\{(a, b) \in H_1 : h_r(a, b) = 0\}) \\ &= \{\chi^{-1}(a, b) \in \tilde{H}_1 : h_r(a, b) = 0\} \\ &= \{(a, b) \in \tilde{H}_1 : h_r \circ \chi(a, b) = 0\}. \end{aligned}$$

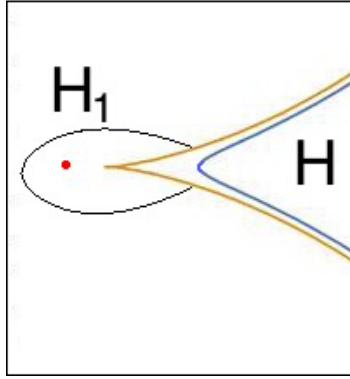


Figure 4.3: The outer yellow curve indicates part of $\text{Per}_1(1) \cap \{a = \bar{b}\}$, and the inner blue curve (along with the red point) indicates part of the deformation $\text{Per}_1(r) \cap \{a = \bar{b}\}$ for some $r \in (1 - \varepsilon, 1)$. The cusp point c_0 on the yellow curve is a critical point of h_1 , i.e. a singular point of $\text{Per}_1(1)$, and the red point is a critical point of h_r ; i.e. a singular point of $\text{Per}_1(r)$.

Clearly, $\chi^{-1}(\alpha_r)$ lies on the curve $\text{Per}_n(1)$; i.e. $\tilde{h}_r(\chi^{-1}(\alpha_r)) = 0$. We claim that $\chi^{-1}(\alpha_r)$ is a non-degenerate critical point of $\tilde{h}_r$, for each $r \in (1 -$

$\varepsilon, 1)$. Recall that $\alpha_r \rightarrow (-\frac{3}{4}, -\frac{3}{4})$ as $r \rightarrow 1$, so $\chi^{-1}(\alpha_r) \rightarrow (\bar{c}_0, c_0)$ as $r \rightarrow 1$. Since $\tilde{h}_r$ (for $\varepsilon > 0$ small enough) is an arbitrarily small perturbation of $\tilde{h}_1$, this claim implies that any small perturbation $\tilde{h}_r$ of $\tilde{h}_1$ has at least one non-degenerate critical point near $(\bar{c}_0, c_0)$. Hence, the Milnor number $\mu(\tilde{h}_1, (\bar{c}_0, c_0)) \geq 1$ (compare [Wal04, Lemma 6.5.5]); i.e. $(\bar{c}_0, c_0)$ is a singular point of $\text{Per}_n(1)$.

To conclude the proof, we only need to prove the claim that $\chi^{-1}(\alpha_r)$ is a non-degenerate critical point of $\tilde{h}_r$; i.e. it has Milnor number $\mu(\tilde{h}_r, \chi^{-1}(\alpha_r)) = 1$. Note that since $\alpha_r \in H_1$ is a non-degenerate critical point of h_r and χ is a biholomorphism from $\tilde{H}_1$ onto H_1 , it follows (by a simple computation using chain rule) that $\chi^{-1}(\alpha_r) \in \tilde{H}_1$ is a non-degenerate critical point of $h_r \circ \chi$. Therefore, the Milnor number $\mu(h_r \circ \chi, \chi^{-1}(\alpha_r)) = 1$. It now suffices to look at the relation between $\tilde{h}_r$ and $h_r \circ \chi$ locally near $\chi^{-1}(\alpha_r)$. This is a routine exercise in analytic geometry, we work out the details for the sake of completeness.

Consider the ring $\mathcal{O}_{2, \chi^{-1}(\alpha_r)}$ of germs of holomorphic functions (in two variables) defined in some neighborhood of $\chi^{-1}(\alpha_r)$. Since $\tilde{h}_r$ and $h_r \circ \chi$ are both square-free as elements of $\mathcal{O}_{2, \chi^{-1}(\alpha_r)}$ and their vanishing define the same analytic set germ near $\chi^{-1}(\alpha_r)$ (namely the germ of $\text{Per}_n(r)$ at $\chi^{-1}(\alpha_r)$), it follows that there exists an invertible element $u \in \mathcal{O}_{2, \chi^{-1}(\alpha_r)}$ such that $\tilde{h}_r = u(h_r \circ \chi)$ as elements of $\mathcal{O}_{2, \chi^{-1}(\alpha_r)}$. But then, their Milnor numbers are equal; i.e. $\mu(h_r \circ \chi, \chi^{-1}(\alpha_r)) = 1 = \mu(\tilde{h}_r, \chi^{-1}(\alpha_r))$. $\square$

On the other hand, we observed that the curve $\text{Per}_1(1)$ is non-singular everywhere along the parabolic arcs of period 1; i.e. $(h_1)'(\bar{c}, c) \neq 0 \ \forall c \in \mathcal{C}$. Consequently, for $r \in (1 - \varepsilon, 1)$, the dynamically defined deformations h_r have no critical points near $\mathcal{C}$. Since this property is preserved under the straightening map χ , and since the curve germs $\chi^{-1}(\text{Per}_1(r))$ form a 1-parameter family of deformations of $\text{Per}_n(1)$, we have the following proposition:

Proposition 4.1.7. *Let $\mathcal{C}$ be a parabolic arc of odd period n of $\mathcal{M}_2^*$, and $c : \mathbb{R} \rightarrow \mathcal{C}$ be its critical Ecalle height parametrization. Then $\forall c \in \mathcal{C}$, $(\bar{c}, c)$ is a non-singular point of $\text{Per}_n(1)$. In particular, the critical Ecalle height parametrization of $\mathcal{C}$ has a non-vanishing derivative at all points; i.e. $c'(t) \neq 0 \ \forall t \in \mathbb{R}$.*

4.2 Real-Analyticity of Hausdorff Dimension

In this final section, we prove real-analyticity of Hausdorff dimension of the Julia sets along the parabolic arcs making use of certain real-analyticity results from [SU14] and applying them to the analytic family of parabolic maps constructed in the previous section. Recall that a rational map is called parabolic if it has at least one parabolic cycle and every critical point of the map lies

in the Fatou set (i.e. is attracted to an attracting or a parabolic cycle). We begin with some known properties of Hausdorff dimension and related objects in the realm of parabolic rational maps. The following concept of radial Julia sets was introduced by Urbański and McMullen [Urb03, McM00].

Definition 4.2.1 (Radial Julia Set). *A point $z \in J(f)$ is called a radial point if there exists $\delta > 0$ and an infinite sequence of positive integers $\{n_k\}$ such that there exists a univalent inverse branch of f^{n_k} defined on $B(f^{n_k}(z), \delta)$ sending $f^{n_k}(z)$ to z for all k . The set of all radial points of $J(f)$ is called the radial Julia set and is denoted as $J_r(f)$. Equivalently, the radial Julia set can be defined as the set of points in $J(f)$ whose ω -limit set non-trivially intersects the complement of the post-critical closure.*

The dynamics at the radial points of a Julia set have a certain expansion property which makes these points easier to study. Indeed, for a radial point z , there exists a sequence of iterates $\{f^{n_k}(z)\}$ accumulating at a point w outside the post-critical closure and hence, there exists a sequence of univalent inverse branches of f^{n_k} defined on some $B(w, \varepsilon)$ sending $f^{n_k}(z)$ to z for all k . Such a sequence necessarily forms a normal family and any limit function must be a constant map (compare [Bea91, Theorem 9.2.1, Lemma 9.2.2]). This shows that the sequence of univalent inverse branches of f^{n_k} are eventually contracting; in other words, $\lim_{n_k \rightarrow \infty} (f^{n_k})'(z) = \infty$.

Conformal measures have played a crucial role in the study of the dimension-theoretic properties of rational maps. It is worth mentioning in this regard that conformal measures satisfy a weak form of Ahlfors-regularity at the radial points of a Julia set. More precisely, by an immediate application of Koebe's distortion theorem and the expansion property at the radial points discussed above, one obtains that for every point z in $J_r(f)$, there exists a sequence of radii $\{r_k(z)\}_{k=1}^\infty$ converging to 0 ($r_k(z) \approx \delta |f^{n_k}(z)|^{-1}$) such that if m is a t -conformal measure for f , then

$$C^{-1} < \frac{m(B(z, r_k(z)))}{r_k(z)^t} < C$$

for some constant C depending on δ and m .

For parabolic rational maps, one has a rather simple but useful description of the radial Julia set. The following proposition was proved in [DU91], we include the proof largely for the sake of completeness.

Lemma 4.2.2. *Let f be a parabolic rational map and let Ω be the set of all parabolic periodic points of f . Then, $J_r(f) = J(f) \setminus \bigcup_{i=0}^{\infty} f^{-i}(\Omega)$. In particular, $\text{HD}(J_r(f)) = \text{HD}(J(f))$.*

Proof. For a parabolic rational map, the post-critical closure intersects the Julia set precisely in Ω , which is a finite set consisting of (parabolic) periodic

points. Starting with any point of $\bigcup_{i=0}^{\infty} f^{-i}(\Omega)$, one eventually lands on Ω under the dynamics and the existence of infinitely many post-critical points in every neighborhood of Ω obstructs the existence of infinitely many univalent inverse branches. It follows that $\bigcup_{i=0}^{\infty} f^{-i}(\Omega) \subset J(f) \setminus J_r(f)$.

It remains to prove the reverse inclusion. By passing to an iterate, one can assume that $f^{\circ q}(z_j) = z_j \forall z_j \in \Omega$. By the description of the local dynamics near a parabolic point [Mil06, §10], there is a neighborhood $B(\Omega, \vartheta')$ of the parabolic points that is contained in the union of the attracting and repelling petals. By continuity, there exists a $0 < \vartheta < \vartheta'$ such that $f^{\circ q}(B(z_i, \vartheta)) \cap B(z_j, \vartheta) = \emptyset$ for $z_i \neq z_j$ and $z_i, z_j \in \Omega$. We claim that for any $z \in J(f) \setminus \bigcup_{i=0}^{\infty} f^{-i}(\Omega)$, there exists a sequence of positive integers $\{n_k\}$ such that the sequence $\{f^{\circ q n_k}(z)\}$ lies outside $B(\Omega, \vartheta)$. Otherwise, there would exist an $n_0 \in \mathbb{N}$ such that $f^{\circ q n}(z) \in B(\Omega, \vartheta) \forall n > n_0$. Then, there exists some $z_j \in \Omega$ such that $f^{\circ q n}(z) \in B(z_j, \vartheta) \forall n > n_0$. But since each $f^{\circ q n}(z)$ belongs to the repelling petal, it follows that $f^{\circ q n_0}(z) = z_j$, a contradiction to the assumption that $z \notin \bigcup_{i=0}^{\infty} f^{-i}(\Omega)$. Finally, observe that any point in $J(f) \setminus B(\Omega, \vartheta)$ has a small neighborhood disjoint from the post-critical set. Since $J(f) \setminus B(\Omega, \vartheta)$ is a compact metric space, there exists a $\delta > 0$ such that the neighborhood $B(w, \delta)$ of any point w of $J(f) \setminus B(\Omega, \vartheta)$ is disjoint from the post-critical set. Therefore, the balls $B(f^{\circ q n_k}(z), \delta)$ are disjoint from the post-critical closure and hence, there exists a univalent inverse branch of $f^{\circ q n_k}$ defined on $B(f^{\circ q n_k}(z), \delta)$ sending $f^{\circ q n_k}(z)$ to z for all k . This proves that $J(f) \setminus \bigcup_{i=0}^{\infty} f^{-i}(\Omega) \subset J_r(f)$.

The final assertion directly follows as the set $\bigcup_{i=0}^{\infty} f^{-i}(\Omega)$ is countable. $\square$

Before we state the main technical theorem from [SU14], we need a couple of more definitions and facts from thermodynamic formalism. We will assume familiarity with the basic definitions and properties of topological pressure [Wal79], [Wal82, §9]. The behavior of the pressure function $t \mapsto P(t) = P(f|_{J(f)}, -t \log |f'|)$, $t \in \mathbb{R}$ (where $P(f|_{J(f)}, -t \log |f'|)$, $t \in \mathbb{R}$ is the topological pressure) has been extensively studied by many people in the context of rational and transcendental maps. In particular, much is known for hyperbolic and parabolic rational maps.

Theorem 4.2.3. [DU91] *For a parabolic rational map f ,*

1. *The function $t \mapsto P(t)$ is continuous, non-increasing and non-negative.*
2. *$\exists s > 0$ such that,*

- (a) $P(t) > 0$ for $t \in [0, s)$,
- (b) $P(t) = 0$ for $t \in [s, \infty)$,
- (c) $P|_{[0, s]}$ is injective.

The following important theorem relates the Hausdorff dimension of the Julia set, the minimal zero of the pressure function and the minimum exponent of t -conformal measures for parabolic rational maps.

Theorem 4.2.4. [DU91] *For a parabolic rational map f , the following holds:*

$$\begin{aligned} \text{HD}(J(f)) &= \inf\{t \in \mathbb{R} : \exists t\text{-conformal measure for } f|_{J(f)}\} \\ &= \inf\{t \in \mathbb{R} : P(t) = 0\} \end{aligned}$$

A more elaborate account of these lines of ideas can be found in the expository article of Urbański [DU91, Urb03].

Definition 4.2.5. *A meromorphic function $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ is called tame if its post-singular set does not contain its Julia set.*

Clearly, parabolic polynomials are tame. For tame rational maps, there exist nice sets [RL07, Dob11] giving rise to conformal iterated function systems with the property that the Hausdorff dimension of the radial Julia set is equal to the common value of the Hausdorff dimensions of the limit sets of all the iterated function systems induced by all nice sets (compare [SU14, §2,3]). One can define the pressure function for these iterated function systems (induced by the nice sets) and the system S is called strongly N -regular if there is $t \geq 0$ such that $0 < P_S(t) < +\infty$ and if there is $t \geq 0$ such that $P_S(t) = 0$. The fact that the conformal iterated function systems arising from parabolic rational maps satisfy this property, can be proved as in Theorem 4.2.3.

We are now prepared to state the main result from [SU14] which is at the technical heart of our real-analyticity result.

Theorem 4.2.6. [SU14, Theorem 1.1] *Assume that a tame meromorphic function $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ is strongly N -regular. Let $\Lambda \subset \mathbb{C}^d$ be an open set and let $\{f_\lambda\}_{\lambda \in \Lambda}$ be an analytic family of meromorphic functions such that*

1. $f_{\lambda_0} = f$ for some $\lambda_0 \in \Lambda$,
2. *there exists a holomorphic motion $H : \Lambda \times J_{\lambda_0} \rightarrow \mathbb{C}$ such that each map H_λ is a topological conjugacy between f_{λ_0} and f_λ on J_{λ_0} .*

Then the map $\Lambda \ni \lambda \mapsto \text{HD}(J_r(f_\lambda))$ is real-analytic on some neighborhood of λ_0 .

Proof of Theorem 1.2.4. Let $\mathcal{C}$ be a parabolic arc and $c : \mathbb{R} \rightarrow \mathcal{C}$ be its critical Ecalle height parametrization. It follows from Lemma 4.1.3 that there exists

an injective holomorphic map $F : S \rightarrow \mathbb{C}^2, w \mapsto (a(w), b(w))$ and an analytic family of q.c. maps $(\varphi_w)_{w \in S}$ with $\varphi_w \circ P_{a(0), b(0)} \circ \varphi_w^{-1} = P_{a(w), b(w)}$. Setting $\Lambda = S$, and $H = \varphi(w, z) := \varphi_w(z)$, we see that all the conditions of Theorem 4.2.6 are satisfied and hence, the function $w \mapsto \text{HD}(J_r(P_{a(w), b(w)}))$ is real-analytic. Restricting the map to the reals, we conclude that the map $h \mapsto \text{HD}(J_r(f_{c(h)}))$ is real-analytic. The result now follows from Lemma 4.2.2. $\square$

Remark 4.2.7. *The techniques used in this chapter can be generalized to prove corresponding statements about the real-analyticity of Hausdorff dimension of the Julia sets on the curves $\text{Per}_n(1)$, under suitable conditions. In particular, the real-analyticity of $\text{HD}(J(f))$ continues to hold on regions of parameter spaces where the maps only have attracting or parabolic cycles and such that all the critical points converge to these cycles. The parabolic arcs of the multicorns inherit the real-analyticity property from the connectedness loci of $\mathcal{F}_d$, which is a sub-family of all polynomials of degree d^2 .*

Chapter 5

Non-Landing Parameter Rays

It is well known that every parameter ray at a periodic angle of the multibrot sets lands at a single parabolic parameter [GM93, Theorem C.7]. The proof of this result uses the fact that there are only finitely many parabolic parameters with given combinatorics in $\mathcal{M}_d$.

Let us recall the structure of the boundaries of the hyperbolic components of odd period of the multicorns to analyze the accumulation properties of the corresponding parameter rays. Let H be a hyperbolic component of odd period $k \neq 1$ (with center c_0) of the multicorn $\mathcal{M}_d^*$. The first return map of the closure of the characteristic Fatou component of c_0 fixes exactly $d+1$ points on its boundary. Only one of these fixed points disconnect the Julia set, and is the landing point of two distinct dynamical rays at $2k$ periodic angles. Let the set of the angles of these two rays be $S' = \{\alpha_1, \alpha_2\}$. Each of the remaining d fixed points is the landing point of precisely one dynamical ray at a k periodic angle; let the collection of the angles of these rays be $S = \{\vartheta_1, \vartheta_2, \dots, \vartheta_d\}$. We can, possibly after renumbering, assume that $0 < \alpha_1 < \vartheta_1 < \vartheta_2 < \dots < \vartheta_d < \alpha_2$ and $\alpha_2 - \alpha_1 < \frac{1}{d}$.

By Theorem 1.2.2 and its proof, ∂H is a simple closed curve consisting of $d+1$ parabolic arcs, and the same number of cusp points such that every arc has two cusp points at its ends. Exactly 1 of these $d+1$ parabolic arcs is a root arc, and the parameter rays at angles α_1 and α_2 accumulate on this arc. The rest of the d parabolic arcs are co-root arcs, and each of them contains the accumulation set of exactly one $\mathcal{R}_{\vartheta_i}^d$. Furthermore, by Lemma 3.3.6 and Lemma 3.3.8, any two parameters on the same parabolic arc on ∂H are combinatorially indistinguishable. This heuristically suggests that a parameter ray accumulating on a parabolic arc of $\mathcal{M}_d^*$ has no reason to land; indeed, there is no combinatorially ‘preferred’ parameter on such an arc.

The goal of the present chapter is to give a complete description of the accumulation properties of the parameter rays of the multicorns. We confirm the above heuristics in Theorem 1.2.6. In Theorem 5.3.2, we give an algorithm to check whether a given rational parameter ray $\mathcal{R}_t^d$ lands or not, in terms of

the combinatorics of the angle t .

The results of this chapter were proved, and are being published jointly with Hiroyuki Inou. A preprint of this publication is available on arXiv [IM14].

5.1 Parabolic Implosion and Horn Maps

The main technical tool used in the proof of the non-trivial accumulation of parameter rays is the perturbation of antiholomorphic parabolic points. This section will be devoted to a brief recollection of the concepts of near-parabolic antiholomorphic Fatou coordinates, and the transit map. Our discussion will roughly follow [HS14, §4]. The technique of perturbation of antiholomorphic parabolic points will allow us to transfer information from the dynamical planes to the parameter plane, and will be an important ingredient in our proof of umbilical cord wiggling in Chapter 7. A more general account on the theory of perturbation of parabolic points can be found in [Dou94, Shi00].

Throughout this section, we will assume that c is a non-cusp parabolic parameter of odd period k on the parabolic arc $\mathcal{C}$ on the boundary of the hyperbolic component H . All of our discussions will concern the characteristic parabolic point of f_c . We will denote an attracting petal of f_c by V_c^{in} and a repelling petal of f_c by V_c^{out} . There exists an open neighborhood U of c (in the parameter plane) such that for all $c' \in U^- := U \setminus \overline{H}$, the characteristic parabolic point splits into two simple periodic points and the Fatou coordinates persist throughout U^- . More precisely, for $c' \in U^-$, there exist an incoming domain $V_{c'}^{\text{in}}$ and an outgoing domain $V_{c'}^{\text{out}}$ having the two simple periodic points on their boundaries. There exists a curve joining the two simple periodic points, which we call the “gate”, such that the points in the incoming domain eventually transit through the gate, and escape to the outgoing domain (as shown in Figure 5.1). Furthermore, there exist injective holomorphic maps $\psi_{c'}^{\text{in}} : V_{c'}^{\text{in}} \rightarrow \mathbb{C}$ and $\psi_{c'}^{\text{out}} : V_{c'}^{\text{out}} \rightarrow \mathbb{C}$ such that $\psi_{c'}^{\text{in}}(f_{c'}^{\circ 2k}(z)) = \psi_{c'}^{\text{in}}(z) + 1$, whenever z and $f_{c'}^{\circ 2k}(z)$ are both in $V_{c'}^{\text{in}}$ (and the same is true for $\psi_{c'}^{\text{out}}$). Since the map $f_{c'}^{\circ k}$ commutes with $f_{c'}^{\circ 2k}$, it induces antiholomorphic self-maps from $C_{c'}^{\text{in}}$ (respectively $C_{c'}^{\text{out}}$) to itself. As $f_{c'}^{\circ k}$ interchanges the two periodic points at the ends of the gate, it interchanges the ends of the cylinders, so it must fix a closed geodesic in the cylinders $\mathbb{C}/\mathbb{Z}$. This is similar to the situation at the parabolic parameter c , so we will call this invariant geodesic the equator. As for the parabolic parameter c , we will choose our Fatou coordinates such that they map the equators to the real line. Thus we can again define Ecalle height as the imaginary part in these Fatou coordinates. We will denote the Ecalle height of a point $z \in C_{c'}^{\text{in/out}}$ by $E(z)$.

We need to normalize our Fatou coordinates (recall that after we decide that the Fatou coordinate maps the equator to the real line, we are left with one real additive degree of freedom of the Fatou coordinate). We can choose two continuous functions $a, b : \overline{U^-} \rightarrow \mathbb{C}$ such that $a(c')$ (respectively $b(c')$) lies

on the incoming (respectively outgoing) equator in $V_{c'}^{\text{in}} \setminus V_{c'}^{\text{out}}$ (respectively in $V_{c'}^{\text{out}} \setminus V_{c'}^{\text{in}}$), for all $c' \in \overline{U^-}$. Let us normalize $\psi_{c'}^{\text{in}}$ and $\psi_{c'}^{\text{out}}$ by the requirements $\psi_{c'}^{\text{in}}(a(c')) = 0$ and $\psi_{c'}^{\text{out}}(b(c')) = 0$. With these normalization, we have that $(c', z) \mapsto \psi_{c'}^{\text{in}}(z)$ and $(c', z) \mapsto \psi_{c'}^{\text{out}}(z)$ are continuous functions on the open sets $\mathcal{V}^{\text{in}} := \{(c', z) : z \in V_{c'}^{\text{in}}\}$ and $\mathcal{V}^{\text{out}} := \{(c', z) : z \in V_{c'}^{\text{out}}\}$ (respectively) in $(\overline{U^-}) \times \mathbb{C}$ (for $c' \in \overline{H}$, $\psi_{c'}^{\text{in}}$ and $\psi_{c'}^{\text{out}}$ are respectively the attracting and repelling Fatou coordinates for $f_{c'}$) [Dou94, §17].

It follows that for every $c' \in U^-$, the quotients $C_{c'}^{\text{in}} := V_{c'}^{\text{in}}/f_{c'}^{\circ 2k}$ and $C_{c'}^{\text{out}} := V_{c'}^{\text{out}}/f_{c'}^{\circ 2k}$ (the quotients of $V_{c'}^{\text{in}}$ and $V_{c'}^{\text{out}}$ by the dynamics, identifying points that are on the same finite orbits entirely in $V_{c'}^{\text{in}}$ or in $V_{c'}^{\text{out}}$) are complex annuli isomorphic to $\mathbb{C}/\mathbb{Z}$. The isomorphisms are given by Fatou coordinates which depend continuously on the parameter throughout U^- .

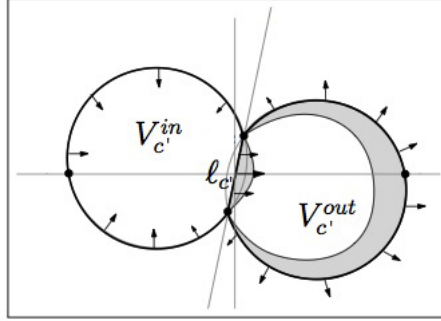


Figure 5.1: The typical dynamical picture after perturbation of the parabolic point (Figure courtesy Dierk Schleicher).

For $c' \in U^-$, the incoming and the outgoing cylinders are isomorphic to each other by a natural biholomorphism, namely $f_{c'}^{\circ 2k}$. This isomorphism is called the “Transit map” and is denoted by $T_{c'}$. The transit map clearly depends continuously on the parameter $c' \in U^-$. It maps the fixed geodesic of the incoming cylinder to the fixed geodesic of the outgoing cylinder and preserves the upper (respectively lower) ends of the cylinders. Thus it must preserve Ecalle heights. The existence of this special isomorphism allows us to relate the Ecalle heights of points in the incoming and the outgoing cylinders and is one of the principal tools in our study.

Finally, we are in a position to state the key technical lemma that helps us to transfer dynamical information at a parabolic parameter to the parameter plane. One can define the disjoint unions :

$$C^{\text{in}} = \bigsqcup_{c' \in \overline{U^-}} C_{c'}^{\text{in}}, \text{ and } C^{\text{out}} = \bigsqcup_{c' \in \overline{U^-}} C_{c'}^{\text{out}}.$$

which are topologically trivial bundles with fibers isomorphic to $\mathbb{C}/\mathbb{Z}$. Note that $\mathbb{C}/\mathbb{Z}$ is equipped with a dynamically marked circle $\mathbb{R}/\mathbb{Z}$, which corresponds to the image of the equator under the Fatou coordinate. Choose a smooth curve $s \mapsto c(s)$ in $\overline{U}$ (in parameter space), parametrized by $s \in [0, \delta]$ for some $\delta > 0$, with $c(0) = c$ and $c(s) \in U^-$ for $s > 0$. Choose a smooth curve $s \mapsto \zeta(s)$ (in the dynamical planes, typically the critical value), also defined for $s \in [0, \delta]$ such that $\zeta(s) \in V_{c(s)}^{\text{in}}$ for all $s \in [0, \delta]$. Then $s \mapsto \zeta(s)$ induces a map $\sigma : [0, \delta] \rightarrow C^{\text{in}}$ with $\sigma(s) \in C_{c(s)}^{\text{in}}$. The following was proved in [HS14, Proposition 4.8].

Lemma 5.1.1 (Limit of Perturbed Fatou Coordinates). *The curve $\gamma := s \mapsto T_{c(s)}(\sigma(s))$ in C^{out} , parametrized by $s \in (0, \delta]$, spirals as $s \downarrow 0$ towards the circle in C_c^{out} at Ecalle height $E(\sigma(0))$.*

Before giving a formal proof of the lemma, let us explain its intuitive meaning. Since C^{out} is a trivial topological bundle of bi-infinite annuli $\mathbb{C}/\mathbb{Z}$ over $\overline{U^-}$; we can choose a trivialization

$$\Theta : C^{\text{out}} \rightarrow \overline{U^-} \times \mathbb{C}/\mathbb{Z}$$

by requiring that $b(c')$ corresponds to the origin of $\mathbb{C}/\mathbb{Z}$. This allows us to define the *phase* to be a continuous lift $\varphi : (0, \delta] \rightarrow \mathbb{R}$ of

$$(0, \delta] \ni s \mapsto \Re(\pi_2(\Theta(\gamma(s)))) \in \mathbb{R}/\mathbb{Z}.$$

As s tends to 0, i.e. as we march closer to the parabolic parameter c , $\zeta(s) \in V_{c(s)}^{\text{in}}$ in the dynamical plane of $c(s)$ takes larger and larger number of iterates to pass through the gate. In other words, the number of fundamental domains that $\zeta(s)$ has to cross in order to escape through the gate tends to $+\infty$, as s tends to 0. This implies that the *phase* $\varphi(s)$ goes to $-\infty$ as s tends to 0, and the image of γ accumulates exactly on the circle on C_c^{out} at Ecalle height $E(\sigma(0))$.

Proof. We first need to choose a continuous lift $\varphi : (0, \delta] \rightarrow \mathbb{R}$ of $(0, \delta] \ni s \mapsto \Re(\pi_2(\Theta(\gamma(s)))) \in \mathbb{R}/\mathbb{Z}$. For $s \in (0, \delta]$, let N_s be the least integer such that $\alpha(s) := f_{c(s)}^{\circ 2kN_s}(\zeta(s))$ lies in $V_{c(s)}^{\text{out}}$ and satisfies $\Re(\psi_{c(s)}^{\text{out}}(\alpha(s))) \geq 0$. For any fixed n , $f_{c(s)}^{\circ 2kn}(\zeta(s)) \rightarrow f_c^{\circ 2kn}(\zeta(0))$ as $s \rightarrow 0$. But in the dynamical plane of f_c , the forward orbit of $\zeta(0)$ under $f_c^{\circ 2k}$ forever lives in the attracting petal $V_{c(0)}^{\text{in}}$. Since $\alpha(s) \in V_{c(s)}^{\text{out}} \setminus V_{c(s)}^{\text{in}}$, this implies that $N_s \rightarrow +\infty$ as $s \rightarrow 0$.

We can now choose φ to be

$$(0, \delta] \ni s \mapsto \Re(\psi_{c(s)}^{\text{out}}(\alpha(s))) - N_s.$$

In fact, $s \mapsto \Re(\psi_{c(s)}^{\text{out}}(\alpha(s)))$ has infinitely many jump discontinuities, but subtracting N_s makes it continuous.

By our normalization of Fatou coordinates, $\Re(\psi_{c(s)}^{\text{out}}(\alpha(s))) \in [0, 1)$. It now follows that $\varphi(s) \rightarrow -\infty$ as $s \rightarrow 0$.

The fact that the image of γ accumulates exactly on the circle on C_c^{out} at Ecalle height $E(\sigma(0))$ follows from the continuity of the Fatou coordinates (in particular, from the continuity of Ecalle height) and the fact that the transit map preserves Ecalle heights. $\square$

We will conclude this section with an analysis of some special properties of the horn maps in the antiholomorphic setting. Once again, we exploit the symmetry between the upper and lower ends of the Ecalle cylinders provided by the antiholomorphic return map. For the sake of completeness, we include the basic definitions and properties of the horn maps. More comprehensive accounts on these ideas can be found in [BE02, §2].

The characteristic parabolic point z_c (say) of f_c has exactly two petals, one attracting and one repelling (denoted by $\mathcal{P}^{\text{att}}$ and $\mathcal{P}^{\text{rep}}$ respectively). The intersection of the two petals has two connected components. We denote by $\mathcal{U}^+$ the connected component of $\mathcal{P}^{\text{att}} \cap \mathcal{P}^{\text{rep}}$ whose image under the Fatou coordinates is contained in the upper half-plane and by $\mathcal{U}^-$ the one whose image under the Fatou coordinates is contained in the lower half-plane. We define the “sepals” $\mathcal{S}^\pm$ by

$$\mathcal{S}^\pm = \bigcup_{n \in \mathbb{Z}} f_c^{\circ 2nk}(\mathcal{U}^\pm)$$

Note that each sepal contains a connected component of the intersection of the attracting and the repelling petals and they are invariant under the first holomorphic return map of the parabolic point. The attracting Fatou coordinate ψ^{att} (respectively the repelling Fatou coordinate ψ^{rep}) can be extended to $\mathcal{P}^{\text{att}} \cup \mathcal{S}^+ \cup \mathcal{S}^-$ (respectively to $\mathcal{P}^{\text{rep}} \cup \mathcal{S}^+ \cup \mathcal{S}^-$) such that they conjugate the first holomorphic return map to the translation $\zeta \mapsto \zeta + 1$.

Definition 5.1.2 (Lifted horn maps). *Let us define $V^- = \psi^{\text{rep}}(\mathcal{S}^-)$, $V^+ = \psi^{\text{rep}}(\mathcal{S}^+)$, $W^- = \psi^{\text{att}}(\mathcal{S}^-)$ and $W^+ = \psi^{\text{att}}(\mathcal{S}^+)$. Then, denote by $H_c^- : V^- \rightarrow W^-$ the restriction of $\psi^{\text{att}} \circ (\psi^{\text{rep}})^{-1}$ to V^- and by $H_c^+ : V^+ \rightarrow W^+$ the restriction of $\psi^{\text{att}} \circ (\psi^{\text{rep}})^{-1}$ to V^+ . We refer to $H_c^\pm$ as lifted horn maps for f_c at z_c .*

The regions $V^\pm$ and $W^\pm$ are invariant under translation by 1. Moreover, the asymptotic development of the Fatou coordinates implies that the regions V^+ and W^+ contain an upper half-plane, whereas the regions V^- and W^- contain a lower half-plane. Consequently, under the projection $\pi : \zeta \mapsto w = \exp(2i\pi\zeta)$, the regions V^+ and W^+ project to punctured neighborhoods $\mathcal{V}^+$ and $\mathcal{W}^+$ of 0, whereas V^- and W^- project to punctured neighborhoods $\mathcal{V}^-$ and $\mathcal{W}^-$ of ∞ .

The lifted horn maps $H_c^\pm$ satisfy $H_c^\pm(\zeta + 1) = H_c^\pm(\zeta) + 1$ on $V^\pm$. Thus, they project to mappings $h_c^\pm : \mathcal{V}^\pm \rightarrow \mathcal{W}^\pm$ such that the following diagram commutes:

$$\begin{array}{ccc} V^\pm & \xrightarrow{H_c^\pm} & W^\pm \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{V}^\pm & \xrightarrow{h_c^\pm} & \mathcal{W}^\pm \end{array}$$

It is well-known that $\exists \eta_c, \eta'_c \in \mathbb{C}$ such that $H_c^+(\zeta) \approx \zeta + \eta_c$ when $\Im(\zeta) \rightarrow +\infty$ and $H_c^-(\zeta) \approx \zeta + \eta'_c$ when $\Im(\zeta) \rightarrow -\infty$. This proves that $h_c^+(w) \rightarrow 0$ as $w \rightarrow 0$. Thus, h_c^+ extends analytically to 0 by $h_c^+(0) = 0$. One can show similarly that h_c^- extends analytically to ∞ by $h_c^-(\infty) = \infty$.

Definition 5.1.3 (Horn Maps). *The maps $h_c^+ : \mathcal{V}^+ \cup \{0\} \rightarrow \mathcal{W}^+ \cup \{0\}$ and $h_c^- : \mathcal{V}^- \cup \{\infty\} \rightarrow \mathcal{W}^- \cup \{\infty\}$ are called horn maps for f_c at z_c .*

Observe that the constants η_c and η'_c are, in general, not well-defined as they depend on particular normalizations of the Fatou coordinates. However, in the antiholomorphic situation, we can and will choose the normalizations of Fatou coordinates described in Lemma 1.1.14 and these Fatou coordinates conjugate the first (antiholomorphic) return map in both petals to $\zeta \mapsto \bar{\zeta} + 1/2$. This choice involves an adjustment of the vertical degree of freedom of the Fatou coordinates. Consequently, the two lifted horn maps H_c^+ and H_c^- are conjugated to each other by $\zeta \mapsto \bar{\zeta} + \frac{1}{2}$. It follows that $\eta'_c = \bar{\eta}_c$. Note that with the chosen normalizations of the Fatou coordinates, the imaginary parts of η_c and η'_c (which are the asymptotic vertical translation constants of the lifted horn maps) become well-defined real numbers. We study the asymptotic behavior of η_c as c tends to the ends of the parabolic arc $\mathcal{C}$.

Lemma 5.1.4. $\Im(\eta_c) \rightarrow +\infty$ as c tends to the ends of the parabolic arc $\mathcal{C}$.

Proof. It follows from the symmetry of the two lifted horn maps that the two horn maps h_c^+ and h_c^- which are defined respectively in neighborhoods of 0 and of ∞ are conjugated by $w \mapsto -1/\bar{w}$ and they asymptotically look like $w \mapsto \exp(2\pi i \eta_c)w$ and $w \mapsto \exp(2\pi i \bar{\eta}_c)w$ respectively. Clearly, $(h_c^+)'(0) = \exp(2\pi i \eta_c)$ and $(h_c^-)'(\infty) = \exp(-2\pi i \bar{\eta}_c)$. By [BE02, Proposition 1], $\exp(-4\pi \Im(\eta_c)) = (h_c^+)'(0)(h_c^-)'(\infty) = \exp(4\pi^2(1 - \iota_c))$, where ι_c is the holomorphic fixed-point index of f_c^{o2k} at the parabolic fixed point z_c . Towards the ends of a parabolic arc, the fixed-point index ι_c at the characteristic parabolic point tends to $+\infty$ in $\mathbb{R}$ ([HS14, Proposition 3.7]). Hence, $\Im(\eta_c) \rightarrow +\infty$ towards the ends of a parabolic arc. $\square$

5.2 Wiggling of Parameter Rays

The goal of this section is to prove Theorem 1.2.6. We begin with an easy lemma.

Lemma 5.2.1. *The parameter rays $\{\mathcal{R}_t^d : t = 0, \frac{1}{d+1}, \frac{2}{d+1}, \dots, \frac{d}{d+1}\}$ of $\mathcal{M}_d^*$ land.*

Proof. Let $\omega = \exp(\frac{2\pi i}{d+1})$. The anti-polynomials f_c and $f_{\omega c}$ are conformally conjugate via the linear map $z \mapsto \omega z$. It follows that $\mathcal{M}_d^*$ has a $(d+1)$ -fold rotational symmetry and $f_c \sim f_{\omega c} \sim f_{\omega^2 c} \sim \dots \sim f_{\omega^d c}$. Also, $K(f_{\omega^j c}) = \omega^j K(f_c)$ and $J(f_{\omega^j c}) = \omega^j J(f_c)$. The Böttcher maps are related by $\omega^j \varphi_c(z) = \varphi_{\omega^j c}(\omega^j z)$. This reads, in terms of external rays, as $\omega^j R_c(\vartheta) = R_{\omega^j c}(\vartheta + \frac{j}{d+1})$.

The map $\Phi : \mathbb{C} \setminus \mathcal{M}_d^* \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$, defined by $c \mapsto \varphi_c(c)$ (where φ_c is the Böttcher coordinate near ∞) is a real-analytic diffeomorphism. This map defines the parameter rays of the multicorns. It follows that $\Phi(\omega^j c) = \omega^j \Phi(c)$.

Now,

$$\begin{aligned} \mathcal{R}_0^d &= \{c \in \mathbb{C} : \Phi(c) = r, r > 1\} \\ &= \{c \in \mathbb{C} : \frac{1}{\omega^j} \Phi(\omega^j c) = r, r > 1\} \\ &= \{c \in \mathbb{C} : \Phi(\omega^j c) = r \exp\left(\frac{2\pi i j}{d+1}\right), r > 1\} \\ &= \{c \in \mathbb{C} : \omega^j c \in \mathcal{R}_{\frac{j}{d+1}}^d\} \\ &= \frac{1}{\omega^j} \mathcal{R}_{\frac{j}{d+1}}^d. \end{aligned}$$

Hence, $\omega^j \mathcal{R}_0^d = \mathcal{R}_{\frac{j}{d+1}}^d$. Since $\mathcal{R}_0^d \subset \mathbb{R}$, it lands. It follows that $\mathcal{R}_{\frac{j}{d+1}}^d$, being the image of $\mathcal{R}_0^d$ under a rotation, must land as well. $\square$

Having taken care of the parameter rays at fixed angles, we now turn our attention to the parameter rays that accumulate on the boundaries of hyperbolic components of odd period *greater than* 1. The proof of Theorem 1.2.6 is carried out in various steps. Let us sketch the key ideas of the proof to stop the readers from getting lost in the technicalities. We will stick to the terminologies used in this chapter so far.

Let $t \in S \cup S'$ and $\mathcal{C}$ be the parabolic arc where the parameter ray $\mathcal{R}_t^d$ accumulates. For every parameter on $\mathcal{C}$, the dynamical ray at angle t lands at the characteristic parabolic point through the unique repelling petal. We first show that if for some $c \in \mathcal{C}$, the dynamical ray $R_c(t)$ projects to a horizontal line under the repelling Fatou coordinate, then the rational lamination of f_c must be invariant under a certain affine transformation. This is achieved by considering a pair of dynamically meaningful involutions in the repelling petal.

A simple combinatorial exercise then shows that such invariant laminations can never exist when the period of H is greater than 1. This proves that for every parameter on $\mathcal{C}$, the projection of the dynamical ray $R_c(t)$ under the repelling Fatou coordinate must traverse a non-degenerate interval of Ecalle heights. The final part of the proof involves a careful parabolic perturbation argument which allows us to transfer the variation of Ecalle heights of the dynamical rays at angle t to the wiggling of the corresponding ray in the parameter plane.

We denote the repelling Fatou coordinate at the characteristic parabolic point of f_c by $\psi^{\text{rep}} : \mathcal{P}^{\text{rep}} \rightarrow \mathbb{H}_{\text{Left}}$ and the Böttcher coordinate by $\varphi_c : \mathcal{A}_\infty(f_c) \rightarrow \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Lemma 5.2.2 (Invariance of Lamination). *Let $c \in \mathcal{C}$ where $\mathcal{C}$ is a parabolic arc on the boundary of H . Suppose that the dynamical ray $R_c(t)$ projects to a horizontal line under the repelling Fatou coordinate. Then the rational lamination $\mathcal{RL}(f_c)$ is invariant under the transformation $s \mapsto 2t - s$.*

Remark 5.2.3. *The projection of the basin of infinity onto the repelling Ecalle cylinder is a conformal annulus bounded by fractal structures (the Julia set and the decorations thereof) from above and below and the projection of the dynamical ray is the unique simple closed geodesic (the core curve) of this conformal annulus. The core curve is a round circle if and only if the annulus is symmetric with respect to this round circle and heuristically speaking, this is an extremely unlikely situation for a polynomial. This lemma essentially tells that such a miracle could happen only if the polynomial had some strong global symmetry.*

Proof. Let L be the horizontal line which is the image of the dynamical ray $R_c(t)$ under the repelling Fatou coordinate. We denote the reflection in $\mathbb{H}_{\text{Left}}$ w.r.t. L as ι_1 . This gives a local antiholomorphic diffeomorphism $((\psi^{\text{rep}})^{-1} \circ \iota_1 \circ \psi^{\text{rep}})$ in the domain of definition of the repelling Fatou coordinate. On the other hand, consider the reflection in the Böttcher coordinate with respect to the radial line at angle t (denoted by ι_2). This gives an antiholomorphic diffeomorphism $(\varphi_c^{-1} \circ \iota_2 \circ \varphi_c)$ in the basin of infinity $\mathcal{A}_\infty(f_c)$, preserving $R_c(t)$ and mapping a dynamical ray $R_c(s)$ to $R_c(2t - s)$ (see Figure 5.2). We will first show that these two diffeomorphisms agree on $\mathcal{P}^{\text{rep}} \cap \mathcal{A}_\infty(f_c)$.

Let $\psi^{\text{rep}} \circ \varphi_c^{-1}$ be defined on a domain D which we can assume to be symmetric w.r.t. the radial line at angle t (under the map ι_2). Since $\psi^{\text{rep}} \circ \varphi_c^{-1}$ maps the radial line at angle t to a horizontal line in the right half-plane, the Schwarz reflection principle implies that $\psi^{\text{rep}} \circ \varphi_c^{-1}(w) = \iota_1 \circ \psi^{\text{rep}} \circ \varphi_c^{-1} \circ \iota_2(w) \forall w \in D$. This implies that $\varphi_c^{-1} \circ \iota_2 \circ \varphi_c = (\psi^{\text{rep}})^{-1} \circ \iota_1 \circ \psi^{\text{rep}}$ on $\mathcal{P}^{\text{rep}} \cap \mathcal{A}_\infty(f_c)$ (alternatively, note that the two antiholomorphic maps $\varphi_c^{-1} \circ \iota_2 \circ \varphi_c$ and $(\psi^{\text{rep}})^{-1} \circ \iota_1 \circ \psi^{\text{rep}}$ agree on the dynamical ray $R_c(t)$, and hence by the identity principle, they must agree on $\mathcal{P}^{\text{rep}} \cap \mathcal{A}_\infty(f_c)$). Hence, the local

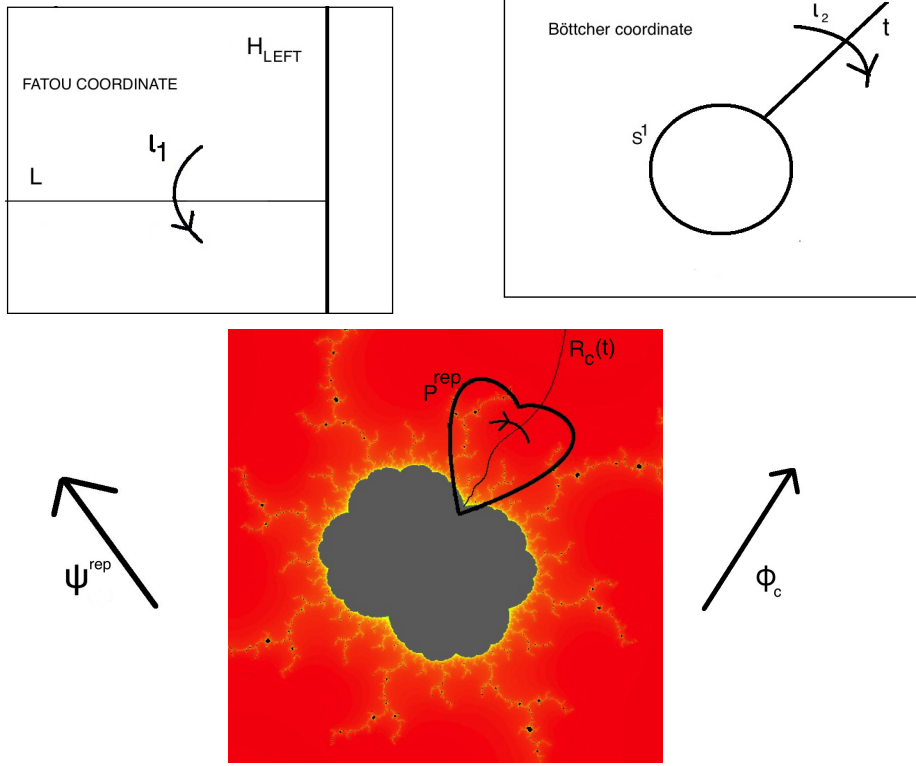


Figure 5.2: Top: The reflections with respect to the straight line L (respectively the radial line at angle t) defined in the left-half plane (respectively in the exterior of the closed unit disk). Bottom: These reflections transported to the dynamical plane via the Fatou (respectively the Böttcher) coordinates agree on their common domain of definitions, namely on $\mathcal{P}^{\text{rep}} \cap \mathcal{A}_\infty(f_c)$.

antiholomorphic diffeomorphism $((\psi^{\text{rep}})^{-1} \circ \iota_1 \circ \psi^{\text{rep}})$ in the repelling petal maps a co-landing ray pair to another co-landing ray pair. It follows that for rational angles s_1 and s_2 , if $R_c(t + s_1)$ and $R_c(t + s_2)$ land at the same point close to the characteristic parabolic point, then so do $R_c(t - s_1)$ and $R_c(t - s_2)$. This proves the local invariance of the rational lamination under the map $s \mapsto 2t - s$.

We now spread this local invariance to the entire rational lamination. Let the rational dynamical rays $R_c(s_1)$ and $R_c(s_2)$ co-land. By the density of iterated pre-images in the Julia set, there exists a co-landing rational ray pair $R_c(s'_1)$ and $R_c(s'_2)$ such that their common landing point lies in $\mathcal{P}^{\text{rep}}$ and $(-d)^{2mk} s'_1 = s_1$, $(-d)^{2mk} s'_2 = s_2$, for some $m \in \mathbb{N}$. By the local invariance, $R_c(2t - s'_1)$ and $R_c(2t - s'_2)$ co-land. By continuity, $f_c^{\circ 2mk}(R_c(2t - s'_1)) = R_c(2t - s_1)$ and $f_c^{\circ 2mk}(R_c(2t - s'_2)) = R_c(2t - s_2)$ co-land as well. This completes the proof of the lemma. $\square$

The proof of the above lemma does not use any fact specific to antiholomorphic dynamics, and hence, the conclusion of the lemma holds for any polynomial of degree $d \geq 2$ with connected Julia set. In general, one does not expect the rational lamination of a polynomial to be invariant under such an affine transformation. This can be easily seen in our case, which is the content of the following:

Lemma 5.2.4 (No Invariant Lamination). *Let $c \in \mathcal{C} \subset \partial H$. Then the rational lamination $\mathcal{RL}(f_c)$ cannot be invariant under the transformation $s \mapsto 2t - s$.*

Proof. We will assume that the rational lamination $\mathcal{RL}(f_c)$ is invariant under the given transformation and arrive at a contradiction.

Case 1 ($t \in S'$).

Without loss of generality, we assume that $t = \alpha_1$. In the dynamical plane of c , $R_c(\alpha_1)$ and $R_c(\alpha_2)$ land at a common point (namely, at the characteristic parabolic point). By the invariance, the dynamical rays at angles α_1 and $2\alpha_1 - \alpha_2$ must land at a common point as well. This is clearly impossible since exactly two rays land at the characteristic parabolic point.

Case 2 ($t \in S$, $2t \neq \alpha_1 + \alpha_2$).

Note that in the dynamical plane of f_c , the dynamical rays $R_c(\alpha_1)$ and $R_c(\alpha_2)$ land at a common point. By the invariance, the dynamical rays at angles $2t - \alpha_1$ and $2t - \alpha_2$ must land at a common point as well. By our assumption, the rays $R_c(2t - \alpha_1)$ and $R_c(2t - \alpha_2)$ lie in different connected components of $\mathbb{C} \setminus (\overline{R_c(\alpha_1)} \cup \overline{R_c(\alpha_2)})$. This forces four different rays $R_c(\alpha_1)$, $R_c(\alpha_2)$, $R_c(2t - \alpha_1)$ and $R_c(2t - \alpha_2)$ to land at a common point (we have used the fact that $0 < \alpha_1 < t < \alpha_2$ and $\alpha_2 - \alpha_1 < 1/2$), which contradicts Theorem 2.1.8.

Case 3 ($t \in S$, $2t = \alpha_1 + \alpha_2$).

We have to work a little harder in this case. Note that for any $i \in \{0, 1, \dots, d\}$, the parameter ray $\mathcal{R}_{\frac{i}{d+1}}^d$ lands on the parabolic arc $\mathcal{C}_i$ on the boundary of the period 1 hyperbolic component. Define the wake W_i to be the connected component of $\mathbb{C} \setminus \{\mathcal{R}_{\frac{i}{d+1}}^d \cup \mathcal{R}_{\frac{i+1}{d+1}}^d \cup \overline{\mathcal{C}_i} \cup \overline{\mathcal{C}_{i+1}}\}$ not containing 0. Then each parameter in W_i has a repelling periodic orbit admitting the orbit portrait $\mathcal{P}_i = \{\{\frac{i}{d+1}, \frac{i+1}{d+1}\}\}$ such that the dynamical rays at angles $\frac{i}{d+1}$ and $\frac{i+1}{d+1}$ together with their common landing point separate the critical value from the critical point.

It is easy to see that H must be contained in some W_j . In particular, $R_c(\frac{j}{d+1})$ and $R_c(\frac{j+1}{d+1})$ land at the same point (and no other ray lands there) and $\frac{j}{d+1} < t < \frac{j+1}{d+1}$. By the invariance property, the rays $R_c(2t - \frac{j}{d+1})$ and

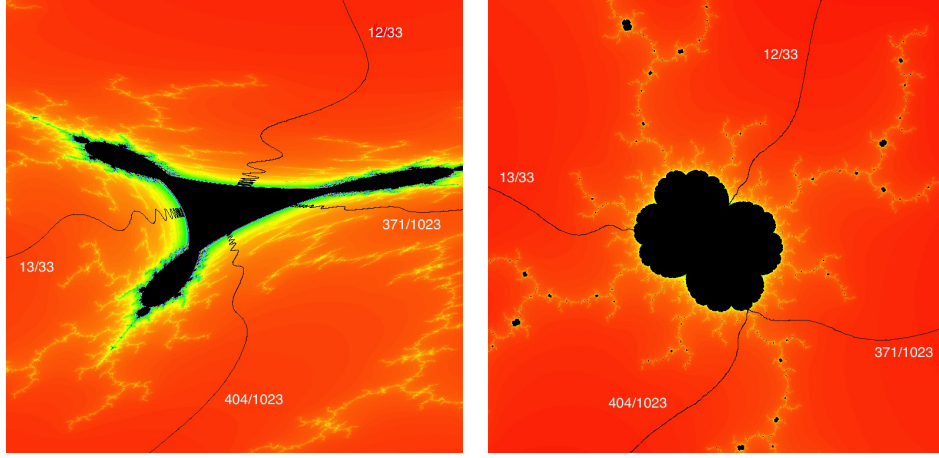


Figure 5.3: Left: Parameter rays accumulating on the boundary of a hyperbolic component of period 5 of the tricorn. Right: The corresponding dynamical rays landing on the boundary of the characteristic Fatou component in the dynamical plane of a parameter on the boundary of the same hyperbolic component.

$R_c(2t - \frac{j+1}{d+1})$ must land at the same point as well. By arguing as in Case (2), we can conclude that $t = \frac{2j+1}{2(d+1)}$. Since t is a periodic angle under multiplication by $-d$, it follows that d must be odd. A simple computation now shows that $(-d)^2 t = t$, which contradicts the fact that the period of t is an odd integer $k \neq 1$. This completes the proof of the lemma. $\square$

Remark 5.2.5. *Case (3) of the previous lemma never occurs for the tricorn. For any hyperbolic component H of odd period $k (\neq 1)$ of the tricorn, we have $S = \{\vartheta_1, \vartheta_2\}$, $S' = \{\alpha_1, \alpha_2\}$ where $(1 + 2^k) \cdot (\vartheta_1 - \alpha_1) = (\alpha_2 - \alpha_1) = (1 + 2^k) \cdot (\alpha_2 - \vartheta_2)$ (Compare Corollary 3.4.18 and Figure 5.3).*

Proof of Theorem 1.2.6. Let $\mathcal{R}_t^d$ accumulates on $\mathcal{C} \subset \partial H$. We will work with the Ecalle height parametrization $c : \mathbb{R} \rightarrow \mathcal{C}$ of $\mathcal{C}$.

Case 1 ($t \in S$).

Let $\tilde{c} = c(0)$ be the parameter on $\mathcal{C}$ whose critical value has (incoming) Ecalle height 0. Note that $\mathcal{C}$ must be a co-root arc of H . By Lemma 5.2.2 and Lemma 5.2.4, the projection of the dynamical ray $R_{\tilde{c}}(t)$ under the repelling Fatou coordinate must traverse a non-degenerate interval of (outgoing) Ecalle heights. Since this dynamical ray is fixed by the first antiholomorphic return map, the interval of (outgoing) Ecalle heights traversed by it must be of the form $[-h, h]$ for some $h > 0$. Since the rays $R_c(t)$ depend uniformly continuously on the parameter c , and since the projection into Ecalle cylinders is

also continuous, we can choose a small neighborhood U of $\tilde{c}$ such that for all $c \in \overline{U} \setminus H$, the projection of the rays $R_c(t)$ into the Ecalle cylinders traverse (outgoing) heights at least $[-h + \varepsilon, h - \varepsilon]$ (Note that in the outgoing cylinder of $\tilde{c}$, $R_{\tilde{c}}(t)$ traverses Ecalle heights $[-h, h]$). To transfer the variation of Ecalle height of $R_c(t)$ to wiggling of the parameter ray $\mathcal{R}_t^d$, we employ Lemma 5.1.1.

We pick a $c(h') \in U$ with $h' \in [-h + 2\varepsilon, h - 2\varepsilon]$ and choose any smooth path $\gamma : [0, \delta] \rightarrow \overline{U}$ with $\gamma(0) = c(h')$ but so that, except for $\gamma(0)$, the path avoids closures of hyperbolic components of period k and so that the path is transverse to $\mathcal{C}$ at $c(h')$.

For $s \in [0, \delta]$, let $z(s)$ be the critical value. For $s > 0$, the critical orbit “transits” from the incoming Ecalle cylinder to the outgoing cylinder; as $s \downarrow 0$, the image of the critical orbit in the outgoing Ecalle cylinder has (outgoing) Ecalle height tending to $h' \in [-h + 2\varepsilon, h - 2\varepsilon]$, while the phase tends to infinity. Therefore, there is $s \in (0, \delta_\varepsilon)$ arbitrarily close to 0 at which the critical value, projected into the incoming cylinder and sent by the transit map to the outgoing cylinder, lands on the projection of the rays $R_{\gamma(s)}(t)$. But in the dynamics of $f_{\gamma(s)}$, this means that the critical value is on one of the dynamical rays $R_{\gamma(s)}(t)$, so $\gamma(s)$ is on the parameter ray $\mathcal{R}_t^d$.

Hence, any smooth path starting at $c(h') \in \mathcal{C} \cap U$ (with $h' \in [-h + 2\varepsilon, h - 2\varepsilon]$) and living inside $\overline{U} \setminus \overline{H}$ thereafter, intersects the parameter ray $\mathcal{R}_t^d$ infinitely often. This proves that $\mathcal{R}_t^d$ cannot land.

Case 2 ($t \in S'$).

Let the interval of (outgoing) Ecalle heights traversed by the dynamical ray $R_{c(h)}(t)$ of $f_{c(h)}$ at angle t be $[l_t(c(h)), u_t(c(h))]$. By Lemma 5.2.2 and Lemma 5.2.4, $u_t(c(h)) > l_t(c(h))$ for every parameter $c(h)$. If we knew that there is a parameter $c(h) \in \mathcal{C}$ with $h \in (l_t(c(h)), u_t(c(h)))$ (observe that this was automatic in Case (1)), then the proof of the wiggling of the parameter ray $\mathcal{R}_t^d$ would proceed exactly as in the previous case. Hence, it suffices to prove the existence of such a parameter $c(h)$.

Consider parameters $c(h)$ with h is sufficiently close to $+\infty$ such that $c(h)$ is a point of a period doubling bifurcation. Perturbing this parameter outside H , we again obtain the ‘open gate’ situation, and for such a perturbed parameter, the critical value belongs to a period $2k$ Fatou component which lies above the corresponding dynamical ray at angle t . Hence, the critical value exits through the gate staying in the period $2k$ Fatou component all along and its Ecalle height (here, we do not need to distinguish between incoming and outgoing heights as the height is preserved in the process of transiting through the gate) is necessarily greater than the minimum Ecalle height of the dynamical t -ray. Since Fatou coordinates vary continuously under perturbation, this shows that $h \geq l_t(c(h))$ for parameters $c(h)$ with h is sufficiently close to $+\infty$. Analogously, for parameters $c(h) \in \mathcal{C}$ with h is sufficiently close to $-\infty$, $h \leq u_t(c(h))$. Once again, these two inequalities together with the fact that

the (incoming) Ecalle height of the critical value as well as the interval of (outgoing) Ecalle heights traversed by the dynamical ray at angle t depend continuously on h imply that there is some parameter $c(h)$ on $\mathcal{C}$ for which $h \in (l_t(c(h)), u_t(c(h)))$.

This completes the proof of the theorem. $\square$

5.3 A Combinatorial Classification

In this section, we will give an algorithm to find whether a rational parameter ray $\mathcal{R}_t^d$ lands or oscillates based only on the combinatorics of t . The following lemma will be useful for this purpose.

Recall that finite collection $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k\}$ of subsets of $\mathbb{Q}/\mathbb{Z}$ satisfying the five properties of Theorem 2.1.8 is called a *formal orbit portrait*.

Lemma 5.3.1. *Let $t \in \mathbb{Q}/\mathbb{Z}$ has period $2k$ under multiplication by $-d$, where k is an odd integer. Consider the collection of finite subsets of $\mathbb{Q}/\mathbb{Z}$ given by $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k\}$, where $\mathcal{A}_1 = \{t, (-d)^k t\}$ and $\mathcal{A}_{i+1} = (-d)\mathcal{A}_i$, $i \pmod k$. Then the parameter ray $\mathcal{R}_t^d$ accumulates on the parabolic root arc of a hyperbolic component of period k if and only if $\mathcal{P}$ satisfies the properties of a formal orbit portrait with characteristic angles t and $(-d)^k t$.*

Proof. If $\mathcal{R}_t^d$ accumulates on a sub-arc of the parabolic root arc of an odd period hyperbolic component, then the period of the hyperbolic component must be k , and the dynamical rays $R_{c_0}(t)$ and $R_{c_0}((-d)^k t)$ co-land at the dynamical root of the characteristic Fatou component of the center c_0 of H . In fact, these are the only rays landing there. It is easy to see that t and $(-d)^k t$ generate the orbit portrait $\mathcal{P}$ and they are also the characteristic angles of the orbit portrait.

The proof the converse is similar to Lemma 3.4.7. For completeness, we work out the details here. Without loss of generality, we can assume that the characteristic arc of $\mathcal{P}$ is $(t, (-d)^k t)$. Note that any accumulation point of a parameter ray at a $2k$ -periodic angle is either a parabolic parameter of odd period k or a parabolic parameter of even period r with $r|2k$ such that the corresponding dynamical ray of period $2k$ lands at the characteristic parabolic point in the dynamical plane of that parameter. Thus the set of accumulation points of $\mathcal{R}_\vartheta^d$, for $\vartheta \in \mathcal{A}_1 \cup \dots \cup \mathcal{A}_k$, is contained in $F = \{\text{The closure of the finitely many root arcs of period } k\} \cup \{\text{The finitely many parabolic parameters of even period and of ray period } 2k\}$.

Consider the connected components U_i of $\mathbb{C} \setminus \left(\bigcup_{\vartheta \in \mathcal{A}_1 \cup \dots \cup \mathcal{A}_k} \mathcal{R}_\vartheta^d \cup F \right)$. There are only finitely many components U_i and they are open. The same rays with angles in $\mathcal{A}_1 \cup \dots \cup \mathcal{A}_k$ land at common points throughout every component U_i .

Let U_1 be the component which contains all parameters c outside $\mathcal{M}_d^*$ with external angle $t(c) \in (t, (-d)^k t)$ (there is such a component as $(t, (-d)^k t)$ does not contain any other angle of $\mathcal{P}$). U_1 must have the two parameter rays $\mathcal{R}_t^d$ and $\mathcal{R}_{(-d)^k t}^d$ on its boundary. By the proof of Theorem 2.2.2, each $c \in U_1 \setminus \mathcal{M}_d^*$ has a repelling periodic orbit admitting the portrait $\mathcal{P}$. If the two parameter rays at angles t and $(-d)^k t$ do not land at a common point or accumulate on a common root arc, then U_1 would contain parameters c outside $\mathcal{M}_d^*$ with $t(c) \notin (t, (-d)^k t)$. It follows from the remark at the end of Section 2.2 that such a parameter can never admit the orbit portrait $\mathcal{P}$, a contradiction. Hence, the parameter rays $\mathcal{R}_t^d$ and $\mathcal{R}_{(-d)^k t}^d$ must land at a common even period parabolic parameter of ray period $2k$ or accumulate on a common root arc of period k of $\mathcal{M}_d^*$. But it is easy to see that if $\mathcal{R}_t^d$ and $\mathcal{R}_{(-d)^k t}^d$ co-land at a parabolic parameter, then the period of its parabolic orbit must be odd, ruling out the first possibility. $\square$

Theorem 5.3.2 (Combinatorial Classification). *Let $t \in \mathbb{Q}/\mathbb{Z}$.*

1) *If the period of t under multiplication by $-d$ is $4k$ for some $k \in \mathbb{N}$, then $\mathcal{R}_t^d$ lands at a parabolic parameter on the boundary of a hyperbolic component of period $4k$.*

2) *If the period of t under multiplication by $-d$ is an odd integer k , then it lands if $k = 1$ and accumulates on a sub-arc (of positive length) of a parabolic co-root arc on the boundary of a hyperbolic component of period k otherwise.*

3) *If the period of t under multiplication by $-d$ is $2k$ for some odd integer k , then it accumulates on a sub-arc (of positive length) of the parabolic root arc of a hyperbolic component of period k if and only if the collection of finite subsets of $\mathbb{Q}/\mathbb{Z}$ given by $\mathcal{P} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k\}$, where $\mathcal{A}_1 = \{t, (-d)^k t\}$ and $\mathcal{A}_{i+1} = (-d)\mathcal{A}_i$, $i \pmod k$, satisfies the properties of a formal orbit portrait with characteristic angles t and $(-d)^k t$. Otherwise, it lands at a parabolic parameter on the boundary of a hyperbolic component of period $2k$.*

4) *If t is strictly pre-periodic under multiplication by $-d$, then $\mathcal{R}_t^d$ lands at a Misiurewicz parameter.*

Proof. 1) See Lemma 3.6.2.

2) By Corollary 3.4.15, every rational parameter ray at an angle t of odd period k lands/accumulates on a sub-arc of a parabolic co-root arc of period k . By Theorem 1.2.6, only the rays at fixed angles land at a single point of a parabolic arc, so the others must accumulate on a sub-arc of positive length.

3) This directly follows from Lemma 3.6.2, Lemma 5.3.1 and Theorem 1.2.6.

4) Arguing as in [Sch00, Theorem 1.1 (3)], one sees that for any limit point c of $\mathcal{R}_t^d$, the critical value c is pre-periodic under f_c with fixed period and pre-period. This implies that all the critical points are strictly pre-periodic (with fixed pre-periods and periods) for the holomorphic polynomial $f_c^{\circ 2}$. Since the

accumulation set of a parameter ray is connected, it now suffices to prove that there are only finitely many parameters with these algebraic data.

In fact, it is not hard to see that there are only finitely pairs of complex numbers (a, b) such that the polynomial $(z^d + a)^d + b$ has strictly pre-periodic critical points with fixed pre-periods and periods. The conditions on the critical points determine a pair of distinct algebraic curves in $\mathbb{C}^2$ and their intersection is contained in the connectedness locus of the family of polynomials of degree d^2 (recall that the Julia set of a polynomial is connected if and only if all the critical orbits are bounded). Since the connectedness locus is compact [BH88], it follows from Bézout's theorem that the two algebraic curves under consideration must intersect at a finite set of points. This shows that there are only finitely many Misiurewicz parameters with fixed period and pre-period in the space of degree d unicritical anti-polynomials. $\square$

Chapter 6

A Non-Equidistribution Theorem

In this chapter, we will investigate the topological structure of $\mathcal{M}_d^*$ near the parabolic arcs of period one. More precisely, in Theorem 1.2.7, we will show that every parabolic arc of period 1 of $\mathcal{M}_d^*$ contains a non-degenerate interval each point of which is accessible from the exterior of $\mathcal{M}_d^*$. This gives an affirmative answer to a question asked in [HS14].

A classical theorem of Levin [Lev90] (also compare the recent works along these lines [DF07, Duj14, Duj09, FG13]) states that the centers of the hyperbolic components of the multibrot sets are equidistributed with respect to their harmonic measures. An easy application of Theorem 1.2.7 shows that the analogous statement is false for the multicorns. We also record that the centers of the hyperbolic components of the multicorns are *not* dense on their boundaries. As a further application, we show that there are no bifurcations near the Ecalle height 0 parameters of the parabolic arcs of $\mathcal{M}_d^*$. This fact was first proved in [HS14, Theorem 7.1], the present proof is somewhat simpler, and almost readily follows from the previous results.

The results of this chapter are being published in a joint paper with Hiroyuki Inou, and the preprint is available on arXiv [IM14].

6.1 Undecorated Sub-arcs

Recall that every parabolic arc has, at both ends, an interval of positive length at which a bifurcation from a hyperbolic component of odd period k to a hyperbolic component of period $2k$ occurs. The decorations attached to these period $2k$ components accumulate on sub-arcs of positive length of the parabolic arc [HS14, Theorem 7.3]. In this section, we will prove that for the parabolic arcs of period 1, the accumulation sets of these decorations do not overlap; they stay at a positive distance away from the Ecalle height 0 parameters.

Proof of Theorem 1.2.7. Every multicorn $\mathcal{M}_d^*$ contains a parabolic arc $\mathcal{C}$ of period 1 intersecting the positive real axis at a unique (non-cusp) parameter $c_d = d^{\frac{d}{1-d}}(d-1)$. Note that f_{c_d} has a unique parabolic fixed point $d^{\frac{1}{1-d}}$ on the real line. Due to the rotational symmetries of the multicorns, it suffices to prove the result for this arc. In the dynamical plane of $f_{c_d} = \bar{z}^d + c_d$, the parabolic fixed point has a unique access through the unique repelling petal and the critical value c_d has incoming Ecalle height 0 (in fact, the incoming and outgoing equators are both contained in the real line and so is the critical value). The projection of the Julia set in the repelling cylinder is a pair of disjoint simple closed curves (the Julia set in this case is simply the boundary of the immediate basin of attraction of the parabolic fixed point) and together they bound a cylinder C of finite modulus. This finite modulus cylinder C is the projection of the basin of infinity in the repelling Ecalle cylinder. We will first show that the cylinder C contains a round cylinder containing the equator.

We choose the repelling Fatou coordinate at the parabolic fixed point so that the equator is mapped to the real line. Since f_{c_d} commutes with complex conjugation, our Fatou coordinates also have the same property. This implies that the upper and the lower components (disjoint simple closed curves) of the projection of the Julia set in the repelling Ecalle cylinder are symmetric with respect to the real line. It follows that both these curves stay at a bounded distance away from the real line; in other words, the projection of the basin of infinity in the repelling Ecalle cylinder contains a round cylinder $\mathbb{S}^1 \times [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Alternatively, it is easy to see that the dynamical ray $R_{c_d}(0)$ (and its image under the repelling Fatou coordinate) is contained in the real line, and hence coincides with the equator in the repelling petal. This shows that the equator is contained in the basin of infinity (compare Figure 6.1). Hence, the projection of the basin of infinity in the repelling Ecalle cylinder contains a horizontal round circle, and thus also contains a round cylinder $\mathbb{S}^1 \times [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$.

The final step is to transfer this round cylinder to an undecorated sub-arc in the parameter plane. It is known that the basin of infinity can not get too small when c_d is perturbed a little bit (compare [Dou94, Theorem 5.1(a)]). Since the critical value and the Fatou coordinates depend continuously on the parameter, we can choose a small neighborhood U of c_d such that for all $c \in \bar{U} \setminus H$, the round cylinder $\mathbb{S}^1 \times [-\varepsilon/2, \varepsilon/2]$ is contained in projection of the basin of infinity into the repelling Ecalle cylinder (note that in the outgoing cylinder of c_d , the round cylinder $\mathbb{S}^1 \times [-\varepsilon, \varepsilon]$ is contained in the projection of the basin of infinity), and such that the critical value of f_c has incoming Ecalle height in $[-\varepsilon/4, \varepsilon/4]$.

We claim that $\bar{U} \setminus \bar{H}$ is contained in the exterior of the multicorns. Indeed, let $c \in \bar{U} \setminus \bar{H}$. In the dynamical plane of f_c , the critical orbit of f_c “transits” from the incoming Ecalle cylinder to the outgoing cylinder and the Ecalle

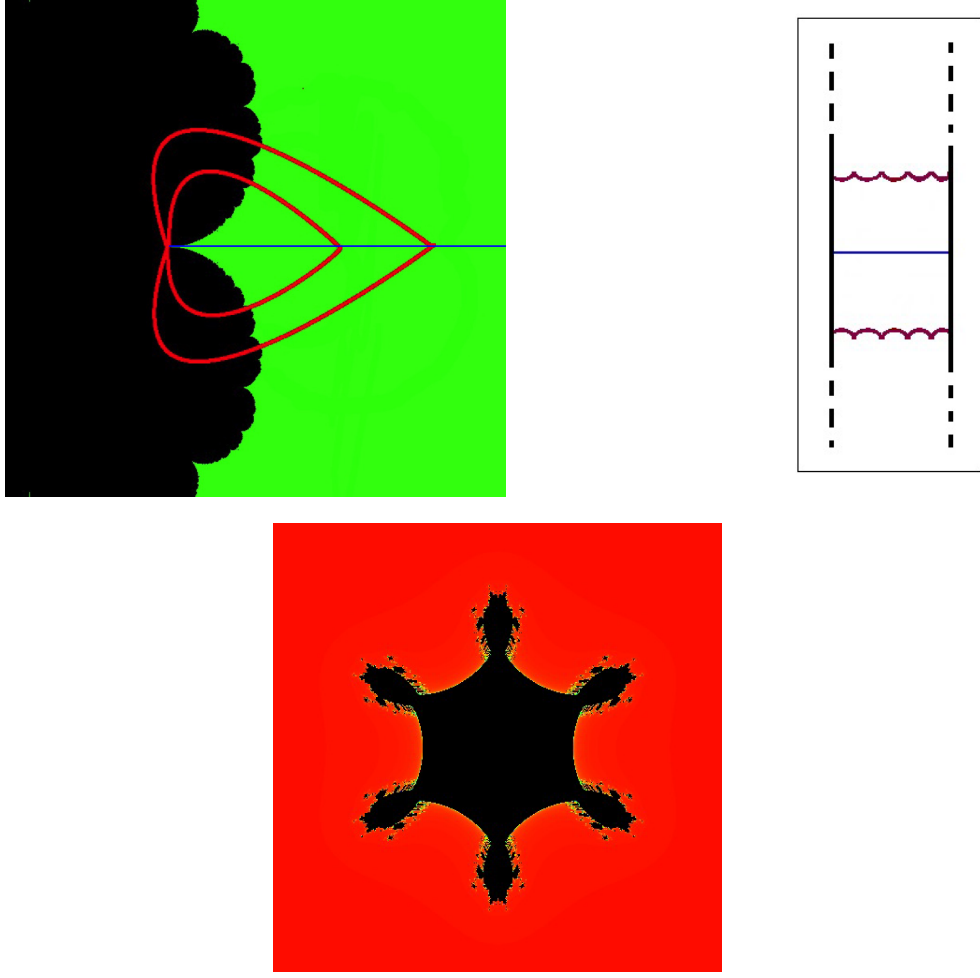


Figure 6.1: Top left: A fundamental domain in the repelling petal under $f_{c_d}^{\circ 2}$. Top right: The image of the fundamental domain under the repelling Fatou coordinate, such that the image of the basin of infinity contains a horizontal line, namely the equator. Bottom: Undecorated sub-arcs on the parabolic arcs of period 1 of $\mathcal{M}_d^*$.

height is preserved in the process. By our construction, this would provide with a point of the critical orbit with outgoing Ecalle height in $[-\varepsilon/4, \varepsilon/4]$ in the repelling Ecalle cylinder. But since $c \in \overline{U} \setminus H$, any point in the repelling cylinder with (outgoing) Ecalle height in $[-\varepsilon/2, \varepsilon/2]$ is contained in the projection of the basin of infinity. Therefore, the critical orbit is contained in the basin of infinity; i.e. $c \notin \mathcal{M}_d^*$.

This implies that $\overline{U} \cap \mathcal{M}_d^* \subset \overline{H}$. Hence, $\mathcal{C}$ contains a sub-arc containing the Ecalle height 0 parameter, no point of which is a limit point of further

decorations; i.e. $\mathcal{C}$ has an undecorated sub-arc. $\square$

Remark 6.1.1. *a) One can prove the following slightly stronger statement for the tricorn: the parabolic arcs of period 1 and 3 contain undecorated sub-arcs. Indeed, in the dynamical plane of a parameter on a parabolic arc of odd period k , the projection of the basin of infinity into the repelling Ecalle cylinder is either an annulus of modulus $\frac{\pi}{2k \ln 2}$ or two disjoint annuli, each of modulus $\frac{\pi}{2k \ln 2}$ (depending on whether the parameter is on a co-root or root arc). For $k = 1$ and 3, this modulus is greater than $1/2$; i.e. the corresponding annuli are not too thin. It is well-known (see [BDH04, Theorem I], for instance) that such a conformal annulus contains a round annulus centered at the origin. In other words, there is an interval I of outgoing Ecalle heights such that in the repelling Ecalle cylinder, the round cylinder $\mathbb{S}^1 \times I$ is contained in the projection of the basin of infinity. One can now prove the existence of undecorated sub-arcs by using the same technique as in Theorem 1.2.7.*

b) Numerical experiments show that away from the real line, the parabolic arcs of sufficiently high periods of the multicorns do not contain undecorated sub-arcs, rather the accumulation sets of the decorations attached to the two bifurcating hyperbolic components at the ends of such an arc overlap. This overlapping phenomenon would automatically make the corresponding parameter rays wiggle on such arcs. However, we do not know how to prove this statement.

6.2 Applications

We finish this chapter with a few interesting consequences of the previous theorem, retaining the terminologies of the previous section.

Corollary 6.2.1. *The centers of hyperbolic components as well as the Misiurewicz parameters are not dense on the boundary of $\mathcal{M}_d^*$.*

Proof. This is obvious from the assertion $\overline{U} \cap \mathcal{M}_d^* \subset \overline{H}$. $\square$

Corollary 6.2.2 (Centers Are Not Equidistributed with Respect to The Harmonic Measure). *Let $A_n = \{c \in \mathbb{C} : f_c^{\circ n}(0) = 0\}$. Then, no sub-sequence of the sequence of measures $\mu_n := \frac{1}{d^{n-1}} \sum_{x \in A_n} \delta_x$ weakly converges to the harmonic measure of $\mathcal{M}_d^*$.*

Proof. It follows from Theorem 1.2.7 (and its proof) that each period 1 parabolic arc $\mathcal{C}$ contains a sub-arc $\mathcal{C}'$ with the following properties:

1. No point of $\mathcal{C}'$ is a limit point of centers of hyperbolic components and hence, $\mathcal{C}'$ is not charged by any sub-sequential limit of $\{\mu_n\}_{n \geq 1}$.

2. Each point of $\mathcal{C}'$ is accessible from the exterior of $\mathcal{M}_d^*$. It is a consequence of a theorem of Lindelöf (see [GM05, Lemma VI.3.1] for a proof) that every point on the boundary of a full compact set in $\mathbb{C}$ that is accessible from its complement, is necessarily the landing point of an external ray (inverse image of a radial line under the Riemann map of the complement of the full compact). Consequently, one can pick any two distinct points on the undecorated arc $\mathcal{C}'$ and these would be the landing points of two distinct external rays (coming from the Riemann map of the complement of $\mathcal{M}_d^*$). It is now easy to see that there exists a non-degenerate interval of angles in $\mathbb{R}/\mathbb{Z}$ such that all external rays at angles in this interval land on $\mathcal{C}'$. Therefore, the harmonic measure of $\mathcal{M}_d^*$ assigns a positive mass to $\mathcal{C}'$.

This completes the proof of the lemma. $\square$

Corollary 6.2.3 (No Bifurcation near Ecalle Height Zero). *On every parabolic arc of period k , the point with Ecalle height zero has a neighborhood (along the arc) that does not intersect the boundary of a hyperbolic component of period $2k$.*

Proof. For the parabolic arcs of period one, the statement readily follows from Theorem 1.2.7. By the proof of Theorem 1.2.6, the Ecalle height 0 parameter on any co-root arc has an open neighborhood (along the arc) which lies in the accumulation set of a parameter ray; hence this neighborhood does not intersect the bifurcating period $2k$ components.

To finish the proof, assume that $\mathcal{C}$ is a root arc. Let α_1 and α_2 be the angles of the parameter rays accumulating on $\mathcal{C}$. In the dynamical plane of any $c \in \mathcal{C}$, the corresponding dynamical rays land at the characteristic parabolic point through two different accesses in the repelling petal. These two accesses are separated by a parabolic Hubbard tree, which is invariant under the first antiholomorphic return map. Clearly, the tree either projects to the equator in the repelling cylinder or its projection traverses an interval of Ecalle heights $[-a, a]$ for some $a > 0$. We can, without loss of generality, assume that the dynamical α_1 (respectively α_2)-ray lies ‘above’ (respectively ‘below’) the hubbard tree (more precisely, this means that the image of the α_1 -ray under the repelling Fatou coordinate lies in a complementary component of the image of the Hubbard tree containing an upper half plane). We denote the interval of Ecalle heights traversed by $R_c(\alpha_1)$ (respectively, $R_c(\alpha_2)$) by $[l_1(c), u_1(c)]$ (respectively, $[l_2(c), u_2(c)]$). It now follows that $u_1(c) > 0$ and $l_2(c) < 0 \forall c \in \mathcal{C}$. Arguing as in case 2 of Theorem 1.2.6, we can find a parameter $c(h) \in \mathcal{C}$ (respectively $c(h')$) with critical Ecalle height $h > 0$ (respectively $h' < 0$) so that $h \in (l_1(c(h)), u_1(c(h)))$ (respectively $h' \in (l_2(c(h')), u_2(c(h')))$). This implies that $c(h)$ and $c(h')$ are in the accumulation sets of the parameter rays at angles α_1 and α_2 respectively. Hence, the convex hull (along $\mathcal{C}$) of the accumulation sets of these two parameter rays contain the Ecalle height

0 parameter on $\mathcal{C}$. This shows that the accumulation sets of two parameter rays bound the Ecalle height 0 parameter on every root arc away from the bifurcating period $2k$ components and completes the proof of the corollary. $\square$

Chapter 7

Discontinuity of The Straightening Map

A standing convention: In the rest of the thesis, we will denote the complex conjugate of a complex number z either by $\bar{z}$ or by z^* . The complex conjugation map will be denoted by ι , i.e. $\iota(z) = z^*$. The image of a set U under complex conjugation will be denoted as $\iota(U)$, and the topological closure of U will be denoted by $\bar{U}$.

Numerical experiments suggest that every odd period hyperbolic component of the multicorns is the basis of a small ‘copy’ of the multicorn itself, much like the Mandelbrot set. While it is true that an antiholomorphic analogue of the straightening theorem does provide us with a map from this small multicorn-like set to the original multicorn, it had been conjectured by various people, including Milnor, Hubbard and Schleicher, that this map is discontinuous [HS14, MP12]. Hiroyuki Inou recently gave a computer-assisted proof of this fact for a particular candidate [Ino14]. We prove this conjecture for every multicorn-like set contained in multicorns of even degree.

The proof of discontinuity is carried out by showing that the straightening map from a baby multicorn-like set to the original multicorn sends certain ‘wiggly’ curves to landing curves. More precisely, for even degree multicorns, there exist hyperbolic components H intersecting the real line, and their ‘umbilical cords’ land on the root parabolic arc on ∂H . In other words, such a component can be connected to the period 1 hyperbolic component by a path. However, we will prove that if H does not intersect the real line or its rotates, then no curve γ contained in $\mathcal{M}_d^* \setminus \bar{H}$ can land on the root parabolic arc on ∂H (this holds for multicorns of any degree). This means that the ‘umbilical cords’ of non-real hyperbolic components always wiggle (see Figure 1.6). Hence, for even degree multicorns, the (inverse of the) straightening map sends a piece of the real line to a ‘wiggly’ curve.

The existence of non-landing umbilical cords for the multicorns was first proved by Hubbard and Schleicher [HS14] (see also [NS96]) under a strong

assumption of non-renormalizability. In order to demonstrate discontinuity of the straightening map, we need to get rid of this hypothesis; i.e. we need to prove wiggling of umbilical cords for all non-real hyperbolic components. The starting point of our proof is the theory of perturbation of antiholomorphic parabolic points as developed in [HS14]. Using these perturbation techniques, they showed that the landing of an umbilical cord at $\tilde{c}$ would imply that the parabolic tree of $f_{\tilde{c}}$ contains a real-analytic arc connecting two bounded Fatou components. With the assumption of non-renormalizability, one can deduce from the above statement that the entire parabolic tree is a real-analytic arc, and this implies that $f_{\tilde{c}}$ and $f_{\tilde{c}^*}$ are conformally conjugate, proving that $\tilde{c}$ lies on the real line (or one of its rotates).

In the general case, we have to adopt a different strategy. From the existence of a small real-analytic arc connecting two bounded Fatou components, we show that the characteristic parabolic germs of $f_{\tilde{c}}$ and $f_{\tilde{c}^*}$ are conformally conjugate. This is the fundamental step in our proof. Since there exists an infinite-dimensional family of conformal conjugacy classes of parabolic germs [Eca75, Vor81]; heuristically speaking, it is extremely unlikely that the parabolic germs of two conformally different polynomials would be conformally conjugate. The next step in our proof involves systematically extending the local analytic conjugacy between parabolic germs to larger domains, step by step. We first extend this local conjugacy to the entire characteristic Fatou component, and then continue it to a neighborhood of the closure of the characteristic Fatou component. This gives us a pair of conformally conjugate polynomial-like restrictions, and applying a theorem of [Ino11], we conclude that some iterates of $f_{\tilde{c}}$ and $f_{\tilde{c}^*}$ are globally conjugate by a finite-to-finite holomorphic correspondence. This means that some iterates of $f_{\tilde{c}}$ and $f_{\tilde{c}^*}$ are (globally) polynomially semi-conjugate to a common polynomial. The final step is to conclude that $\tilde{c}$ is conformally conjugate to a real parameter, by using the theory of decompositions of polynomials with respect to composition, which is due to Ritt [Rit22] and Engstrom [Eng41].

7.1 Conformal Conjugacy of Parabolic Germs

Our goal in this section is to apply the perturbation techniques from Section 5.1 to prove a local consequence of umbilical cord landing. We will work with a fixed hyperbolic component H of odd period $k \neq 1$. Let $\mathcal{C}$ be the root arc on ∂H , $\tilde{c}$ be the Ecalle height 0 parameter on $\mathcal{C}$, and z_1 be the characteristic parabolic point of $f_{\tilde{c}}$. Assume further that the dynamical rays $R_{\tilde{c}}(\vartheta)$ and $R_{\tilde{c}}(\vartheta')$ land at the characteristic parabolic point z_1 . By symmetry, there is a hyperbolic component $\iota(H)$ (which is just the reflection of H with respect to the real line) of the same period k such that $\tilde{c}^*$ is the Ecalle height 0 parameter on the root arc $\iota(\mathcal{C})$ of $\iota(H)$. The characteristic parabolic point of $f_{\tilde{c}^*}$ is z_1^* .

Remark 7.1.1. *As a convention, we will denote the complex conjugate of a*

complex number z by z^* . The complex conjugation map will be denoted by ι , i.e. $\iota(z) = z^*$. The image of a set U under complex conjugation will be denoted as $\iota(U)$, and the topological closure of U will be denoted by $\overline{U}$.

The following lemma essentially states that landing of an umbilical cord at $\tilde{c}$ implies a (local) regularity property of the parabolic tree of $f_{\tilde{c}}$. Although the proof can be extracted from [HS14, Lemma 5.10, Theorem 6.1], we prefer working out the details here as our organization differs from that of [HS14].

Lemma 7.1.2. *If there is a path $p : [0, \delta] \rightarrow \mathbb{C}$ with $p(0) = \tilde{c}$ and $p((0, \delta]) \subset \mathcal{M}_d^* \setminus \overline{H}$, then the repelling equator at z_1 is contained in a loose parabolic tree of $f_{\tilde{c}}$.*

Proof. Since any two bounded Fatou components of $f_{\tilde{c}}$ have disjoint closures, and the inverse images of the characteristic parabolic point z_1 are dense on the Julia set, it follows that any parabolic tree must traverse infinitely many bounded Fatou components, and intersect their boundaries at pre-parabolic points. Furthermore, any loose parabolic tree intersects the Julia set at a Cantor set of points. We first claim that the part of any loose parabolic tree contained in the repelling petal of z_1 intersects the Julia set entirely along the repelling equator. To do this, we will assume the contrary, and will arrive at a contradiction.

If the part of the parabolic tree contained in the repelling petal were not contained in the equator, then there would be a point w_0 (say, repelling pre-periodic) in the intersection having Ecalle height $h > 0$. We construct a sequence (w_n) so that $w_{n+1} := f_{\tilde{c}}^{-\circ 2k}(w_n)$ choosing a local branch of $f_{\tilde{c}}^{-\circ 2k}$ that fixes z_1 , and so that all w_n are in the repelling petal of z_1 . Therefore, $w_n \rightarrow z_1$ as $n \rightarrow \infty$, and all w_n have the same Ecalle height h . Similarly, let $w'_n := f_{\tilde{c}}^{\circ k}(w_n)$, then $w'_n \rightarrow z_1$, and all these points have Ecalle heights $-h$. As w_0 is on the parabolic tree, which is invariant, it follows that w_0 is accessible from outside of the filled-in Julia set on both sides of the tree, so each w_n and w'_n is the landing point of (at least) two dynamic rays, ‘above’ and ‘below’ the tree. If ϑ_n be the angle of dynamical ray landing at w_n from below, then it follows that $\vartheta_n \rightarrow \vartheta$ (say) as $n \rightarrow \infty$, and $R_{\tilde{c}}(\vartheta_n)$ traverses (at least) the interval $[-h/2, h/2]$ of (outgoing) Ecalle heights. Analogously, if ϑ'_n be the angle of dynamical ray landing at w'_n from above, then it follows that $\vartheta'_n \rightarrow \vartheta'$ as $n \rightarrow \infty$, and $R_{\tilde{c}}(\vartheta'_n)$ traverses (at least) the interval $[-h/2, h/2]$ of (outgoing) Ecalle heights.

The dynamical rays at angles ϑ_n and ϑ'_n , and their landing points depend uniformly continuously on the parameter (as they are pre-periodic rays), and the projection of these rays onto the outgoing Ecalle cylinder is also continuous. Hence, there exists a neighborhood U of $\tilde{c}$ in the parameter space such that if $c' \in \overline{U} \setminus H$, then the projection of the dynamical rays $R_{c'}(\vartheta_n)$ and $R_{c'}(\vartheta'_n)$ onto the outgoing cylinder of $f_{c'}^{\circ 2k}$ traverse the interval of (outgoing) Ecalle heights $[-h/3, h/3]$.

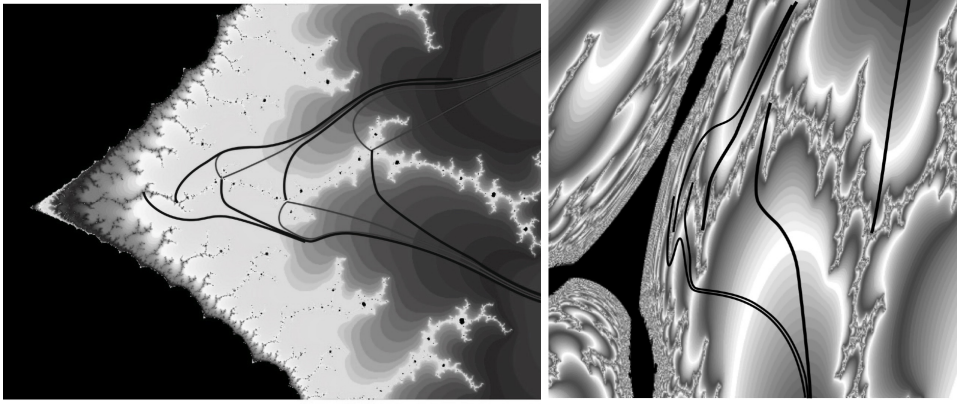


Figure 7.1: Left: Dynamical rays crossing the repelling equator in the dynamical plane. Right: The corresponding parameter rays obstructing the existence of the required path p . (Figures courtesy Dierk Schleicher.)

By assumption, there is a path $p : [0, \delta] \rightarrow \mathbb{C}$ with $p(0) = \tilde{c}$ and $p((0, \delta]) \subset \mathcal{M}_d^* \setminus \overline{H}$. By choosing a smaller δ , we can assume that $p((0, \delta]) \subset \overline{U} \setminus \overline{H}$. For $s > 0$, the critical orbit of $f_{p(s)}$ “transits” from the incoming Ecalle cylinder to the outgoing cylinder; as $s \downarrow 0$, the image of the critical orbit in the outgoing Ecalle cylinder has (outgoing) Ecalle height tending to 0, while the phase tends to infinity (by Lemma 5.1.1). Therefore, there is $s \in (0, \delta)$ arbitrarily close to 0 for which the critical value, projected into the incoming cylinder, and sent by the transit map to the outgoing cylinder, lands on the projection of the dynamical rays $R_{p(s)}(\vartheta_n)$ (or $R_{p(s)}(\vartheta'_n)$). But in the dynamics of $f_{p(s)}$, this means that the critical value is in the basin of infinity, i.e. such a parameter $p(s)$ lies outside $\mathcal{M}_d^*$. This contradicts our assumption that $p((0, \delta]) \subset \mathcal{M}_d^* \setminus \overline{H}$, and proves that the part of any loose parabolic tree contained in the repelling petal of z_1 must intersect the Julia set entirely along the repelling equator.

In fact, the above argument essentially shows that the existence of any dynamical ray (in the repelling petal at z_1) traversing an interval of outgoing Ecalle heights $[-x, x]$ with $x > 0$ would destroy the existence of the required path p (compare Figure 7.1). In other words, for the existence of such a path p , no dynamical ray should ‘cross’ the repelling equator. Therefore, the repelling equator is contained in the filled-in Julia set $K(f_{\tilde{c}})$; i.e. the repelling equator forms the part of a loose parabolic tree. $\square$

So far, we have more or less proceeded in the same direction as in [HS14]. More precisely, we have showed that in order that an umbilical cord lands, the corresponding anti-polynomial must have a loose parabolic tree whose intersection with the repelling petal is an analytic arc (observe that the equator is an analytic arc; i.e. the image of the real line under a biholomorphism).

But without any assumption on non-renormalizability, we *cannot* conclude anything about the global structure of the parabolic tree. To circumvent this problem, we will adopt a different ‘local to global’ principle. The following lemma shows that the local regularity of the parabolic tree, established in the previous lemma, has a very surprising consequence on the corresponding parabolic germs.

Lemma 7.1.3 (Local Analytic Conjugacy of Parabolic Germs). *If the repelling equator of $f_{\tilde{c}}$ at z_1 is contained in a loose parabolic tree, then the parabolic germs given by the restrictions of $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ at their characteristic parabolic points are conformally conjugate by a local biholomorphism that maps $f_{\tilde{c}}^{\circ kr}(\tilde{c})$ to $f_{\tilde{c}^*}^{\circ kr}(\tilde{c}^*)$, for r large enough.*

Proof. Pick any bounded Fatou component U (different from the characteristic Fatou component) that the repelling equator hits. Assume that the equator intersects ∂U at some pre-parabolic point z' . Consider a small piece Γ' of the equator with z' in its interior. Since z' eventually falls on the parabolic orbit, some large iterate of f_c maps z' to z_1 by a local biholomorphism carrying Γ' to an analytic arc Γ (say, $\Gamma = f_{\tilde{c}}^{\circ n}(\Gamma')$) passing through z_1 (compare Figure 7.2). We will show that Γ agrees with the repelling equator (up to truncation). Indeed, the repelling equator and the curve Γ are both parts of two loose parabolic trees (any forward iterate of a loose parabolic tree is again a loose parabolic tree) and hence must coincide along a cantor set of points on the Julia set. As analytic arcs, they must thus coincide up to truncation. In particular, the part of Γ not contained in the characteristic Fatou component is contained in the repelling equator and is forward invariant. Straighten the analytic arc Γ to an interval $(-\varepsilon, \varepsilon) \in \mathbb{R}$ by a local biholomorphism $\alpha : V \rightarrow \mathbb{C}$ such that $z_1 \in V$ and $\alpha(z_1) = 0$ (for convenience, we choose V such that it is symmetric with respect to Γ). This local biholomorphism conjugates the parabolic germ of $f_{\tilde{c}}^{\circ 2k}$ at z_1 to a germ that fixes 0. Moreover, the conjugated germ maps small enough positive reals to positive reals. Clearly, this must be a real germ. Thus, the parabolic germ of $f_{\tilde{c}}^{\circ 2k}$ at z_1 is analytically conjugate to a real germ.

Observe that $\iota : z \mapsto z^*$ is a topological conjugacy between $f_{\tilde{c}}$ and $f_{\tilde{c}^*}$. One can carry out the preceding construction with $f_{\tilde{c}^*}$ and show that the parabolic germ of $f_{\tilde{c}^*}^{\circ 2k}$ at z_1^* is analytically conjugate to a real germ. In fact, the role of Γ' is now played by $\iota(\Gamma')$ and hence, the role of Γ is played by $f_{\tilde{c}^*}^{\circ n}(\iota(\Gamma')) = \iota(\Gamma)$. Then the biholomorphism $\iota \circ \alpha \circ \iota : \iota(V) \rightarrow \mathbb{C}$ straightens $\iota(\Gamma)$. Conjugating the parabolic germ of $f_{\tilde{c}^*}^{\circ 2k}$ at z_1^* by $\iota \circ \alpha \circ \iota$, one recovers the same real germ as in the previous paragraph. Thus, the parabolic germs given by the restrictions of $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ at their characteristic parabolic points are analytically conjugate. Moreover, since the critical orbits of $f_{\tilde{c}}^{\circ 2k}$ lie on the equator, and since the equator is mapped to the real line by α , the conjugacy $\eta := (\iota \circ \alpha \circ \iota)^{-1} \circ \alpha$ preserves the critical orbits; i.e. it maps $f_{\tilde{c}}^{\circ kr}(\tilde{c})$ to $f_{\tilde{c}^*}^{\circ kr}(\tilde{c}^*)$.

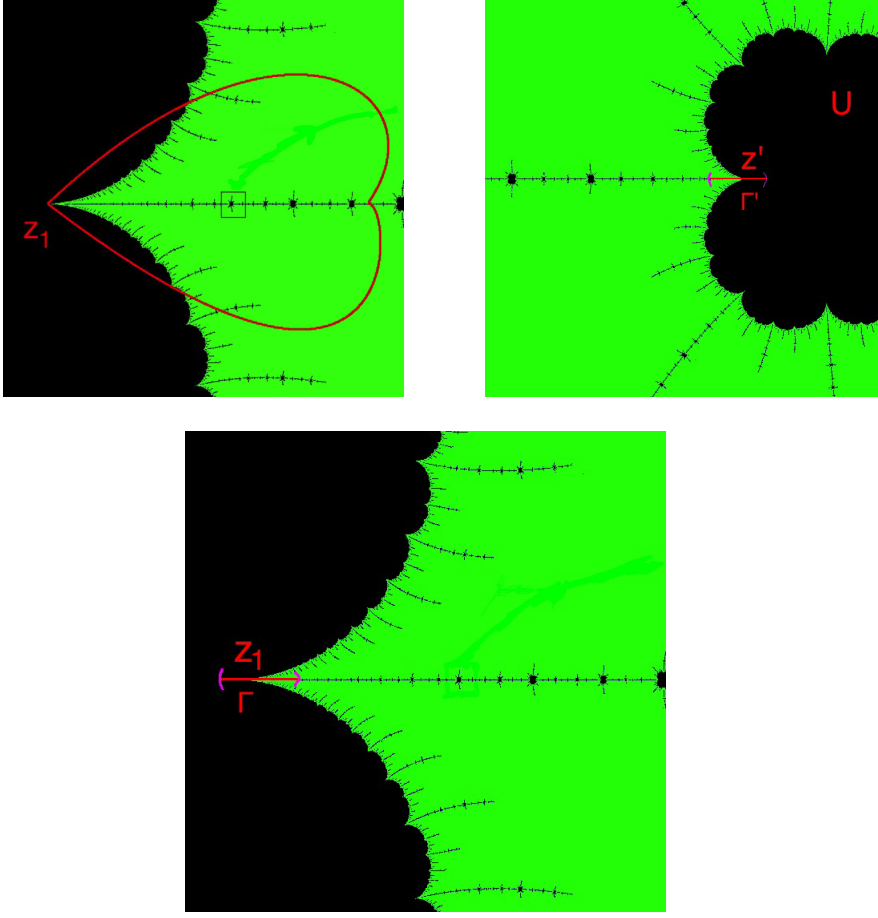


Figure 7.2: Top left: A repelling petal at the characteristic parabolic point is enclosed by the red curve. The repelling equator at the characteristic parabolic point is contained in the filled-in Julia set. The square box contains a Fatou component U such that z' is a point of intersection of ∂U and the repelling equator. Top right: A blow-up of the box shown in the left figure. Bottom: The piece Γ' maps to an invariant analytic arc Γ which passes through the characteristic point z_1 and lies in the filled-in Julia set.

(for r large enough, so that $f_{\tilde{c}}^{\circ kr}(\tilde{c})$ is contained in the domain of definition of α). $\square$

Remark 7.1.4. *It follows from the previous lemma that the real-analytic curve Γ passing through z_1 is invariant under $f_{\tilde{c}}^{\circ k}$. Indeed, Γ is formed by parts of the attracting equator, the repelling equator, and the parabolic point z_1 .*

Remark 7.1.5. *Observe that $f_{\tilde{c}}^{\circ 2k}$ has three critical points and two (infinite) critical orbits in the characteristic Fatou component $U_{\tilde{c}}$. Two of these three*

critical points (of $f_{\tilde{c}}^{\circ 2k}|_{U_{\tilde{c}}}$) are mapped to the same point by $f_{\tilde{c}}^{\circ 2k}$ so that they lie on the same critical orbit of $f_{\tilde{c}}^{\circ 2k}$, and the third one lies on the other critical orbit of $f_{\tilde{c}}^{\circ 2k}$. Hence, the two critical orbits of $f_{\tilde{c}}^{\circ 2k}|_{U_{\tilde{c}}}$ are dynamically distinct.

We want to emphasize the fact that the conjugacy $\eta = (\iota \circ \alpha \circ \iota)^{-1} \circ \alpha$ constructed (from the condition that an umbilical cord lands at $\tilde{c}$) in the previous lemma is special: it maps (the tail of) each of the two (dynamically distinct) critical orbits of $f_{\tilde{c}}^{\circ 2k}$ to (the tail of) the corresponding critical orbit of $f_{\tilde{c}^*}^{\circ 2k}$. In fact, the parabolic germs given by the restrictions of $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ at their characteristic parabolic points are always conformally conjugate by $\iota \circ f_{\tilde{c}}^{\circ k}$; but this local conjugacy exchanges the two post-critical orbits, which have different topological dynamics, and hence this local conjugacy has no chance of being extended to the entire parabolic basin.

7.1.1 A Brief Digression to Parabolic Germs

We have showed that if an umbilical cord lands at $\tilde{c}$, then the restriction of $f_{\tilde{c}}^{\circ 2k}$ at its characteristic parabolic point is analytically conjugate to a real germ. Let us denote the local reflection with respect to the curve Γ (as in the previous lemma) by ι_{Γ} . Then, ι_{Γ} commutes with $f_{\tilde{c}}^{\circ k}$ (near z_1). Therefore locally near z_1 , $f_{\tilde{c}}^{\circ 2k} = (f_{\tilde{c}}^{\circ k} \circ \iota_{\Gamma}) \circ (\iota_{\Gamma} \circ f_{\tilde{c}}^{\circ k}) = g^{\circ 2}$, where $g = f_{\tilde{c}}^{\circ k} \circ \iota_{\Gamma} = \iota_{\Gamma} \circ f_{\tilde{c}}^{\circ k}$. Thus, the parabolic germ given by the restriction of $f_{\tilde{c}}^{\circ 2k}$ in a neighborhood of z_1 is (locally) the second iterate of a holomorphic germ (which is a holomorphic germ with a parabolic fixed point at z_1 and multiplier 1).

By the classical theory of conformal conjugacy classes parabolic germs, one knows that there is an infinite-dimensional family of conformally different parabolic germs [Eca75, Vor81]. With this in mind, the conclusion of Lemma 7.1.3, and the properties of the parabolic germ $f_{\tilde{c}}^{\circ 2k}$ at its characteristic parabolic point discussed in the previous paragraph, seems very unlikely to hold unless the polynomial $f_{\tilde{c}}^{\circ 2k}$ has a strong global symmetry. In fact, in the next section, we will prove that this can happen if and only if $\tilde{c}$ lies on the real line or on one of its rotates. The proof, however, depends heavily on the structure of the polynomial $f_{\tilde{c}}^{\circ 2k}$. But it seems reasonable to try to understand the global implications of a local information about a polynomial parabolic germ, in more general settings. In particular, we ask the following questions:

Question 7.1.6 (From Germs to Polynomials). 1. Let p be a complex polynomial with a parabolic fixed point at 0 with multiplier 1.

- (a) If the parabolic germ $p|_{B(0,\varepsilon)}$ is locally conformally conjugate to a real parabolic germ, or
- (b) if the parabolic germ $p|_{B(0,\varepsilon)}$ is locally the second iterate of a holomorphic germ,

can we conclude that the polynomial p has a corresponding global property?

2. *Let p_1 and p_2 be two complex polynomials with parabolic fixed points at 0 with multiplier 1. If the two parabolic germs p_1 and p_2 at the origin are conformally conjugate, are p_1 and p_2 globally semi-conjugate or semi-conjugate from/to a common polynomial?*

Conditions (1a) and (1b) on the parabolic germ of p translate into corresponding conditions on its extended (lifted) horn map (such that its domain is maximal), i.e. they are real symmetric, or they commute with translation by $1/2$ (see Section 5.1 for the definition of horn maps). This implies that the domains of the extended lifted horn maps are real-symmetric, or invariant under translation by $1/2$. Condition (2) implies that the horn maps of p_1 and p_2 at 0 are in fact the same, up to pre and post composition by multiplications with non-zero complex numbers.

For an explicit polynomial p , it may be possible to check whether the restriction of p in a neighborhood of 0 is locally the second iterate of a holomorphic germ, by looking at the corresponding horn map at 0, say h^- . All one needs to check for this is whether h^- is of the form $\left(\sum_{n \in \mathbb{N}} a_n w^{2n-1}\right)$, $a_n \in \mathbb{C}$.

Indeed, let ψ^{att} and ψ^{rep} be the attracting and repelling Fatou coordinates for p at 0; then these Fatou coordinates will conjugate the local compositional square root of p to the translation $z \mapsto z + 1/2$ (since a local square root commutes with p , it would induce a conformal isomorphism of the Ecalle cylinder whose second iterate is $z \mapsto z + 1$ and the only conformal isomorphism of $\mathbb{C}/\mathbb{Z}$ with this property is $z \mapsto z + 1/2$). Hence, the lifted horn maps $\psi^{\text{att}} \circ (\psi^{\text{rep}})^{-1}$ would commute with $z \mapsto z + 1/2$, and the horn maps would commute with $w \mapsto -w$. Therefore, we would have $h^-(-w) = -h^-(w)$ and hence, h^- must be of the above form. In fact, this necessary condition on the horn map is also sufficient to ensure that the corresponding parabolic germ is locally a second iterate.

7.2 Extending The Local Conjugacy

The main goal of this section is to prove that umbilical cords never land away from the real line or its rotates. More precisely, we will show that the existence of a path as in Lemma 7.1.2 would imply that $\tilde{c} \in \mathbb{R} \cup \omega\mathbb{R} \cup \omega^2\mathbb{R} \cup \dots \cup \omega^d\mathbb{R}$, where $\omega = e^{\frac{2\pi i}{d+1}}$. To this end, we will first extend the local analytic conjugacy between the parabolic germs, constructed in Lemma 7.1.3, to a conformal conjugacy between polynomial-like restrictions. This would allow us to apply a theorem of [Ino11] to deduce that the maps $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ are conjugate by an irreducible holomorphic correspondence. In other words, we will show

that $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ are polynomially semi-conjugate to a common polynomial p . The final step involves proving that $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ are, in fact, affinely conjugate.

Recall that $\tilde{c}$ be the Ecalle height 0 parameter on the root parabolic arc $\mathcal{C}$ of the hyperbolic component H (period k), z_1 its characteristic parabolic point and $U_{\tilde{c}}$ its characteristic Fatou component. We choose a Riemann map $\varphi_{\tilde{c}}$ of $U_{\tilde{c}}$ normalized so that it sends the critical point (of the first return map) to 0, and its homeomorphic extension to the boundary sends the parabolic point on $\partial U_{\tilde{c}}$ to 1. Then, $\varphi_{\tilde{c}}$ conjugates the first holomorphic return map $f_{\tilde{c}}^{\circ 2k}$ on $U_{\tilde{c}}$ to a Blaschke product B . Furthermore, the Ecalle height 0 condition implies that the images of the two critical orbits of $f_{\tilde{c}}^{\circ 2k}$ under an attracting Fatou coordinate are related by translation by $1/2$. It follows that the local compositional square root of $f_{\tilde{c}}^{\circ 2k}$ in an attracting petal (i.e. translation by $1/2$ pulled back by the Fatou coordinate) can be analytically extended throughout the immediate basin $U_{\tilde{c}}$. Therefore, the first return map $f_{\tilde{c}}^{\circ 2k}$ on $U_{\tilde{c}}$ is the second iterate of a holomorphic map g preserving $U_{\tilde{c}}$ with a parabolic fixed point of multiplier $+1$ at z_1 . Since $f_{\tilde{c}}^{\circ 2k}$ has two critical points in $U_{\tilde{c}}$, g is unicritical. Hence, the Riemann map $\varphi_{\tilde{c}}$ conjugates g to the Blaschke product $\tilde{B}(w) = \frac{(d+1)w^d + (d-1)}{(d+1) + (d-1)w^d}$. It follows that $B(w) = \tilde{B}^{\circ 2}(w)$. We record this fact in the following lemma.

Lemma 7.2.1 (Ecalle Height Zero Basins Are Conformally Conjugate). *The first holomorphic return map of an immediate basin of a parabolic point of odd period of a critical Ecalle height 0 parameter is conformally conjugate to $B(w) = \tilde{B}^{\circ 2}(w)$, where $\tilde{B}(w) = \frac{(d+1)w^d + (d-1)}{(d+1) + (d-1)w^d}$. In particular, they are conformally conjugate to each other.*

At this point, we know that $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$, restricted to the characteristic Fatou components, are conformally conjugate, and the corresponding parabolic germs are also conformally conjugate by a ‘critical orbit’-preserving local biholomorphism. The lemma essentially shows that these two conjugacies can be glued together.

Lemma 7.2.2 (Extension to The Immediate Basin). *The conformal conjugacy η between the parabolic germs of $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ at their characteristic parabolic points can be extended to a conformal conjugation between the dynamics in the immediate basins.*

Proof. We need to choose our conformal change of coordinates symmetrically for $\tilde{c}$ and $\tilde{c}^*$.

Let us choose the attracting Fatou coordinate $\psi_{\tilde{c}}^{\text{att}}$ in $U_{\tilde{c}}$ normalized so that it maps the equator to the real line and $\psi_{\tilde{c}}^{\text{att}}(\tilde{c}) = 0$. This naturally determines our preferred attracting Fatou coordinate $\psi_{\tilde{c}^*}^{\text{att}} := \iota \circ \psi_{\tilde{c}}^{\text{att}} \circ \iota$ for $f_{\tilde{c}^*}^{\circ 2k}$ at its characteristic parabolic point z_1^* , and we have $\psi_{\tilde{c}^*}^{\text{att}}(\tilde{c}^*) = 0$.

Recall that we constructed a conformal conjugacy $\eta = \iota \circ \alpha^{-1} \circ \iota \circ \alpha : V \rightarrow \iota(V)$ between the parabolic germs $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ at their characteristic parabolic germs in Lemma 7.1.3. η maps some attracting petal (not containing $\tilde{c}$) $P \subset V$ of $f_{\tilde{c}}^{\circ 2k}$ at z_1 to some attracting petal $\iota(P) \subset \iota(V)$ of $f_{\tilde{c}^*}^{\circ 2k}$ at z_1^* . Hence, $\psi_{\tilde{c}}^{\text{att}} \circ \eta^{-1}$ is an attracting Fatou coordinate for $f_{\tilde{c}^*}^{\circ 2k}$ at z_1^* . By the uniqueness of Fatou coordinates, $\psi_{\tilde{c}}^{\text{att}} \circ \eta^{-1}(z) = \psi_{\tilde{c}^*}^{\text{att}}(z) + a$, for some $a \in \mathbb{C}$ and for all z in their common domain of definition. There is some large n for which $f_{\tilde{c}^*}^{\circ 2kn}(\tilde{c}^*)$ belongs to $\iota(V)$, the domain of definition of η^{-1} . By definition,

$$\begin{aligned} \psi_{\tilde{c}}^{\text{att}} \circ \eta^{-1}(f_{\tilde{c}^*}^{\circ 2kn}(\tilde{c}^*)) &= \psi_{\tilde{c}}^{\text{att}} \circ \alpha^{-1} \circ \iota \circ \alpha \circ \iota \circ f_{\tilde{c}^*}^{\circ 2kn}(\tilde{c}^*) \\ &= \psi_{\tilde{c}}^{\text{att}} \circ \alpha^{-1} \circ \iota \circ \alpha \circ f_{\tilde{c}}^{\circ 2kn} \circ \iota(\tilde{c}^*) \\ &= \psi_{\tilde{c}}^{\text{att}} \left(\alpha^{-1} \left(\iota \left(\alpha \left(f_{\tilde{c}}^{\circ 2kn}(\tilde{c}) \right) \right) \right) \right) \\ &= \psi_{\tilde{c}}^{\text{att}} \left(\alpha^{-1} \left(\alpha \left(f_{\tilde{c}}^{\circ 2kn}(\tilde{c}) \right) \right) \right) \\ &= \psi_{\tilde{c}}^{\text{att}} \left(f_{\tilde{c}}^{\circ 2kn}(\tilde{c}) \right) \\ &= n . \end{aligned}$$

This holds since $\tilde{c}$ lies on the equator, and α maps $f_{\tilde{c}}^{\circ 2nk}(\tilde{c})$ to the real line. But,

$$\begin{aligned} \psi_{\tilde{c}^*}^{\text{att}}(f_{\tilde{c}^*}^{\circ 2kn}(\tilde{c}^*)) &= \iota \circ \psi_{\tilde{c}}^{\text{att}} \circ \iota \left(\left(f_{\tilde{c}}^{\circ 2nk}(\tilde{c}) \right)^* \right) \\ &= n . \end{aligned}$$

This shows that $a = 0$, and hence, $\eta = (\psi_{\tilde{c}^*}^{\text{att}})^{-1} \circ \psi_{\tilde{c}}^{\text{att}}$ on P .

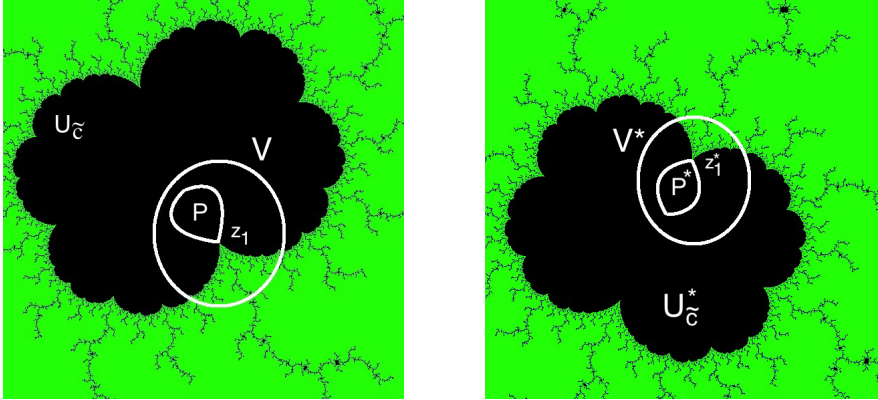


Figure 7.3: The germ conjugacy $\eta : V \rightarrow \iota(V)$ and the basin conjugacy $\chi : U_{\tilde{c}} \rightarrow \iota(U_{\tilde{c}})$ agree with $(\psi_{\tilde{c}^*}^{\text{att}})^{-1} \circ \psi_{\tilde{c}}^{\text{att}} : P \rightarrow \iota(P)$.

We also fix the Riemann map $\varphi_{\tilde{c}} : U_{\tilde{c}} \rightarrow \mathbb{D}$ which conjugates $f_{\tilde{c}}^{\circ 2k}$ on $U_{\tilde{c}}$ to the Blaschke product B in the previous lemma. Since the immediate basin of

$f_{\tilde{c}^*}$ at its characteristic parabolic point z_1^* is $\iota(U_{\tilde{c}})$, $\iota \circ \varphi_{\tilde{c}} \circ \iota$ is the preferred Riemann map of the basin that sends the critical value $\tilde{c}^*$ to 0, sends the parabolic point z_1^* to 1, and conjugates $f_{\tilde{c}^*}^{\circ 2k}$ to the Blaschke product B . A similar argument as above shows that the isomorphism $\chi := \iota \circ \varphi_{\tilde{c}}^{-1} \circ \iota \circ \varphi_{\tilde{c}} : U_{\tilde{c}} \rightarrow \iota(U_{\tilde{c}})$ agrees with $(\psi_{\tilde{c}^*}^{\text{att}})^{-1} \circ \psi_{\tilde{c}}^{\text{att}}$ on P , and hence extends the local conjugacy η to the entire immediate basin $U_{\tilde{c}}$ (compare Figure 7.3), such that it conjugates $f_{\tilde{c}}^{\circ 2k}$ on $U_{\tilde{c}}$ to $f_{\tilde{c}^*}^{\circ 2k}$ on $\iota(U_{\tilde{c}})$. $\square$

Abusing notations, let us denote the extended conjugacy from $U_{\tilde{c}} \cup V$ onto $\iota(U_{\tilde{c}} \cup V)$ of the previous lemma by η . Our next goal is to extend η to a neighborhood of $\overline{U_{\tilde{c}}}$ (the topological closure of $U_{\tilde{c}}$).

Lemma 7.2.3 (Extension to The Closure of The Basin). *η can be extended conformally to a neighborhood of $\overline{U_{\tilde{c}}}$.*

Proof. Observe that the basin boundaries are locally connected, and hence by Caratheodory's theorem, the conformal conjugacy η extends as a homeomorphism from $\partial U_{\tilde{c}}$ onto $\partial \iota(U_{\tilde{c}})$. Moreover, η extends analytically across the point z_1 . At this point, the existence of the required extension follows from [BE07, Lemma 2, Lemma 3]. However, we have a more straightforward proof (essentially using the same idea) as our maps are unbranched on the Julia set.

By Montel's theorem, $\bigcup_n f_{\tilde{c}}^{\circ 2kn}(V \cap \partial U_{\tilde{c}}) = \partial U_{\tilde{c}}$. As none of the $f_{\tilde{c}}^{\circ 2kn}$ have critical points on $\partial U_{\tilde{c}}$, we can extend η in a neighborhood of each point of $\partial U_{\tilde{c}}$ by simply using the equation $\eta \circ f_{\tilde{c}}^{\circ 2kn} = f_{\tilde{c}^*}^{\circ 2kn} \circ \eta$. Since all of these extensions at various points of $\partial U_{\tilde{c}}$ extend the already defined (and conformal) common map η , the uniqueness of analytic continuations yields an analytic extension of η in a neighborhood of $\overline{U_{\tilde{c}}}$. By construction, this extension is clearly a proper holomorphic map, and assumes every point in $\iota(U_{\tilde{c}})$ precisely once. Therefore, the extended η from a neighborhood of $\overline{U_{\tilde{c}}}$ onto a neighborhood of $\iota(U_{\tilde{c}})$ has degree 1, and is a conformal conjugacy between $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$. $\square$

We are now ready to apply the ‘local to global’ result from [Ino11].

Lemma 7.2.4 (Global Semi-conjugacy). *There exist polynomials p , p_1 and p_2 such that $f_{\tilde{c}}^{\circ 2k} \circ p_1 = p_1 \circ p$, $f_{\tilde{c}^*}^{\circ 2k} \circ p_2 = p_2 \circ p$, and $\deg p_1 = \deg p_2$.*

Proof. Note that $f_{\tilde{c}}^{\circ 2k}$ (respectively $f_{\tilde{c}^*}^{\circ 2k}$) restricted to a small neighborhood of $\overline{U_{\tilde{c}}}$ (respectively $\iota(U_{\tilde{c}})$) is polynomial-like of degree 4, and it follows from the previous lemma that these two polynomial-like maps are conformally conjugate. Applying Theorem [Ino11, Theorem 1] to this situation, we obtain the existence of polynomials p , p_1 , and p_2 such that the required semi-conjugacies hold. Since $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ are topologically conjugate by ι , it follows from the proof of Theorem [Ino11, Theorem 1] that $\deg p_1 = \deg p_2$ (observe that the product dynamics $(f_{\tilde{c}}^{\circ 2k}, f_{\tilde{c}^*}^{\circ 2k})$ is globally self-conjugate by $(\iota \times \iota) \circ q$, where $q : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, $q(z, w) = (w, z)$). $\square$

In order to finish the proof of Theorem 1.2.9, we need to use a classification of semi-conjugate polynomials proved in [Ino11, Appendix A]. The results are based on the work of Ritt and Engstrom.

Let $\mathcal{S}$ be the set of all affine conjugacy classes of triples (f, g, h) of polynomials of degree at least two such that $f \circ h = h \circ g$, where we say that two triples (f_1, g_1, h_1) and (f_2, g_2, h_2) are affinely conjugate if there exist affine maps σ_1, σ_2 such that

$$\begin{aligned} f_2 &= \sigma_1 \circ f_1 \circ \sigma_1^{-1}, \\ g_2 &= \sigma_2 \circ g_1 \circ \sigma_2^{-1}, \\ h_2 &= \sigma_1 \circ h_1 \circ \sigma_2^{-1}. \end{aligned}$$

We denote $(f_1, g_1, h_1) \sim (f_2, g_2, h_2)$.

The following theorem tells us that one can always apply a reduction step to assume that $\deg f = \deg g$ and $\deg h$ are coprime.

Theorem 7.2.5. *Let $[(f, g, h)] \in \mathcal{S}$. If $\gcd(\deg f, \deg h) = d > 1$, then there exist polynomials $g_1, h_1, f_1, \hat{h}_1, \alpha_1$ and β_1 such that*

$$\begin{aligned} f \circ h_1 &= h_1 \circ g_1, & f_1 \circ \hat{h}_1 &= \hat{h}_1 \circ g, & h &= h_1 \circ \beta_1 = \alpha_1 \circ \hat{h}_1, \\ \deg f &= \deg g_1 = \deg f_1, & \deg \alpha_1 &= \deg \beta_1 = d, & \deg h_1 &= \deg \hat{h}_1 = \deg h/d. \end{aligned}$$

In particular, if $d < \deg h$, then $[(f, g_1, h_1)], [(f_1, g, \hat{h}_1)] \in \mathcal{S}$.

The next theorem gives a complete classification for the case $\gcd(\deg f, \deg h) = 1$.

Theorem 7.2.6. *Assume that $[(f, g, h)] \in \mathcal{S}$ satisfies $\gcd(\deg f, \deg h) = 1$. Then there exists a representative (f_0, g_0, h_0) of $[(f, g, h)]$ such that one of the following is true:*

- $f(z) = z^r (P(z))^b$, $g(z) = z^r P(z^b)$ and $h(z) = z^b$, where $r = a \bmod b$, and P is a complex polynomial,
- $f = g = T_a$, $h = T_b$ are Chebyshev polynomials (of degree a and b respectively),

where $a = \deg f (= \deg g)$ and $b = \deg h$.

Finally, we have all the ingredients to prove that all non-real umbilical cords wiggle. I am grateful to Hiroyuki Inou for helping me with the application of Engstrom's algebraic reduction step in the proof of this theorem.

Proof of Theorem 1.2.9. Let us quickly recapitulate what we have proved so far. In Lemma 7.1.2 and Lemma 7.1.3, we showed that if there is a path $p : [0, \delta] \rightarrow \mathbb{C}$ with $p(0) = \tilde{c}$ and $p((0, \delta]) \subset \mathcal{M}_d^* \setminus \overline{H}$, then the parabolic germs given by the restrictions of $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ at their characteristic parabolic points

are conformally conjugate by a local biholomorphism that maps $f_{\tilde{c}}^{\circ kr}(\tilde{c})$ to $f_{\tilde{c}^*}^{\circ kr}(\tilde{c}^*)$, for r large enough. Subsequently, in Lemma 7.2.2 and Lemma 7.2.3, we extended this local conformal conjugacy to a conformal conjugacy between two polynomial-like restrictions of $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ respectively. Lemma 7.2.4 provided the existence of two global polynomial semi-conjugacies involving $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$. The rest of the proof is essentially a careful analysis of these semi-conjugacies, from which the conclusion of the theorem will follow.

If the two semi-conjugacies appearing in Lemma 7.2.4 are affine conjugacies (i.e. if p_1 and p_2 have degree 1), then $f_{\tilde{c}}^{\circ 2k}$ and $f_{\tilde{c}^*}^{\circ 2k}$ are affinely conjugate, and a straightforward computation shows that $\tilde{c} \in \mathbb{R} \cup \omega \mathbb{R} \cup \omega^2 \mathbb{R} \cup \dots \cup \omega^d \mathbb{R}$, where $\omega = \exp(\frac{2\pi i}{d+1})$.

Therefore, we only need to deal with the situation $\deg p_1 = \deg p_2 = b > 1$. We will first prove by contradiction that $\gcd(\deg f_{\tilde{c}}^{\circ 2k}, \deg p_1) > 1$. To do this, let $\gcd(\deg f_{\tilde{c}}^{\circ 2k}, \deg p_1) = 1$, i.e. $\gcd(d^{2k}, b) = 1$, i.e. $\gcd(d, b) = 1$. Now we can apply Theorem 7.2.6 to our situation; but since $f_{\tilde{c}}^{\circ 2k}$ is parabolic, it is neither a power map, nor a Chebyshev polynomial. Hence, there exists some non-constant polynomial P such that $f_{\tilde{c}}^{\circ 2k}$ is affinely conjugate to the polynomial $g(z) := z^r(P(z))^b$. If $r \geq 2$, then $g(z)$ has a super-attracting fixed point at 0. But $f_{\tilde{c}}^{\circ 2k}$, which is affinely conjugate to $g(z)$, has no super-attracting fixed point. Hence, $r = 0$ or 1. By degree consideration, we have $d^{2k} = r + bk$, where $\deg P = k$. The assumption $\gcd(d, b) = 1$ implies that $r = 1$, i.e. $g(z) = z(P(z))^b$. Now the fixed point 0 for g satisfies $g^{-1}(0) = \{0\} \cup P^{-1}(0)$, and any point in $P^{-1}(0)$ has a local mapping degree b under g . The same must hold for the affinely conjugate polynomial $f_{\tilde{c}}^{\circ 2k}$: there exists a fixed point (say x) for $f_{\tilde{c}}^{\circ 2k}$ such that any point in $(f_{\tilde{c}}^{\circ 2k})^{-1}(x)$ has mapping degree b except for x ; in particular, all points in $(f_{\tilde{c}}^{\circ 2k})^{-1}(x) \setminus \{x\}$ are critical points for $f_{\tilde{c}}^{\circ 2k}$ (since $b > 1$). However, the local degree of any critical point for $f_{\tilde{c}}^{\circ 2k}$ is equal to d^r for some $r \geq 1$ (every critical point of $f_{\tilde{c}}^{\circ 2}$ has mapping degree d); so $b = d^r$, and this contradicts the assumption that $\gcd(d, b) = 1$ (alternatively, we could use the fact that $f_{\tilde{c}}^{\circ 2k}$ has no finite critical orbit).

Applying Engstrom's theorem [Eng41] (compare [Ino11, Theorem 11, Lemma 13]), we obtain the existence of polynomials (of degree at least two) g , h , g_1 and h_1 such that the following holds, up to affine change of coordinates:

$$f_{\tilde{c}}^{\circ 2k} = h \circ g, \quad f_{\tilde{c}^*}^{\circ 2k} = h_1 \circ g_1, \quad p = g \circ h = g_1 \circ h_1, \quad \text{and } \deg(g) = \deg(g_1).$$

The equation $f_{\tilde{c}}^{\circ 2k} = h \circ g$ implies that $f_{\tilde{c}^*}^{\circ 2k} = (\iota \circ h \circ \iota) \circ (\iota \circ g \circ \iota)$. Note that the only possible non-uniqueness in the decomposition of $f_{\tilde{c}^*}^{\circ 2k}$ into prime factors (under composition) occurs due to the relation

$$z^d + \tilde{c} = (z^{d_1} + \tilde{c}) \circ (z^{d_2}) = (z^{d_2} + \tilde{c}) \circ (z^{d_1}) \quad \text{with } d = d_1 d_2, \quad d_1, d_2 \geq 2.$$

However, we claim that $h_1 = \iota \circ h \circ \iota$ and $g_1 = \iota \circ g \circ \iota$. Indeed, if h_1 and $\iota \circ h \circ \iota$ (and hence g_1 and $\iota \circ g \circ \iota$) have different decompositions, then using the type of non-uniqueness and the relation $p = g \circ h = g_1 \circ h_1$, one obtains two

different sets of multiplicities of the critical points for the same polynomial p . This contradiction proves the claim.

Therefore, $p = g \circ h = (\iota \circ g \circ \iota) \circ (\iota \circ h \circ \iota)$. It now clearly follows that $\tilde{c} \in \mathbb{R}$, up to affine change of coordinates. In other words, $\tilde{c} \in \mathbb{R} \cup \omega \mathbb{R} \cup \omega^2 \mathbb{R} \cup \dots \cup \omega^d \mathbb{R}$, where $\omega = \exp(\frac{2\pi i}{d+1})$. $\square$

Remark 7.2.7. *We should point out that Theorem 1.2.9 shows that if a hyperbolic component H of odd period (different from 1) of $\mathcal{M}_d^*$ does not intersect the real line or its ω -rotates, then H cannot be connected to the principal hyperbolic component (of period 1) by a path inside of $\mathcal{M}_d^*$. This statement is a sharper version of a result of Hubbard and Schleicher on the non path-connectedness of the multicorns [HS14, Theorem 6.2].*

7.3 Renormalization and Multicorn-Like Sets

In this section, we will give a brief overview of the combinatorics and topology of straightening maps in consistence with [IK12]. After preparing the necessary background on renormalization and straightening, we will introduce the concepts of ‘multicorn-like sets’ and the ‘straightening map’ from ‘multicorn-like sets’ to the actual multicorn (compare [Ino14]). Finally, we will state the principal results of [IK12], applied to our setting.

Recall the concepts of anti-polynomial-like maps and their straightening (Definition 1.1.23, Theorem 1.1.25).

Definition 7.3.1 (Renormalization and Straightening). *We say f_c is renormalizable if there exist U'_c, U_c and $k > 1$ such that $f_c^{\circ k} : U'_c \rightarrow U_c$ is unicritical-like, and has a connected filled-in Julia set.*

Such a mapping $f_c^{\circ k} : U'_c \rightarrow U_c$ is called a renormalization of f_c , and k is called its period.

By the straightening theorem, there exists a unique monic centered holomorphic or antiholomorphic unicritical polynomial P hybrid equivalent to $f_c^{\circ k} : U'_c \rightarrow U_c$, up to affine conjugacy. We call P the straightening of the renormalization.

Take $c_0 \in \mathcal{M}_d^*$ such that 0 is a periodic point of period $k > 1$ of f_{c_0} ; i.e. c_0 is a center (of a hyperbolic component of $\text{int}(\mathcal{M}_d^*)$) of period k . Let $\lambda_0 = \lambda(f_{c_0})$ be the rational lamination of f_{c_0} . Define the combinatorial renormalization locus $\mathcal{C}(c_0)$ as follows:

$$\mathcal{C}(c_0) = \{c \in \mathcal{M}_d^* : \lambda(f_c) \supset \lambda_0\}.$$

Since a rational lamination is an equivalence relation on $\mathbb{Q}/\mathbb{Z}$, it is a subset of $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$, and hence, subset inclusion makes sense. By definition, for $c \in \mathcal{C}(c_0)$, the external rays of λ_0 -equivalent angles for f_c land at the same point. Hence those rays divide K_c into ‘fibers’. Let K be the fiber containing

the critical value c . Then $f_c^{\circ k}(K) = K$. We say f_c is c_0 -renormalizable if there exists a (holomorphic or antiholomorphic) unicritical-like restriction $f_c^{\circ k} : U'_c \rightarrow U_c$ such that the filled-in Julia set is equal to K . Let the renormalization locus $\mathcal{R}(c_0)$ with combinatorics λ_0 be:

$$\mathcal{R}(c_0) = \{c \in \mathcal{C}(c_0) : f_c \text{ is } c_0\text{-renormalizable}\}.$$

We call such a renormalization a c_0 -renormalization (see the definition of λ_0 -renormalization in [IK12] for a more general definition). We call k the *renormalization period*.

For the rest of this section, we fix such a c_0 , and its rational lamination $\lambda_0 = \lambda(f_{c_0})$. For $c \in \mathcal{R}(c_0)$, let P be the straightening of a c_0 -renormalization of f_c . By the straightening theorem, P is well-defined. When the renormalization period k is even, then the c_0 -renormalization is holomorphic. Hence, $P = f_{c'}$ for some $c' \in \mathcal{M}_d$. When k is odd, the c_0 -renormalization is anti-holomorphic, so $P = f_{c'}$ for some $c' \in \mathcal{M}_d^*$. In either case, we denote c' by $\chi_{c_0}(c)$ (here, we have tacitly fixed an external marking for our (anti-)polynomial-like maps so that the map χ_{c_0} is well-defined).

To relate our definition of χ_{c_0} with the general notion of straightening for holomorphic polynomials (as developed in [IK12]), we need to work with $P_{\bar{c},c} = f_c^{\circ 2}$. This allows us to embed $\{f_c\}_{c \in \mathbb{C}}$ in the family $\text{Poly}(d^2)$ of monic centered polynomials of degree d^2 . We will denote the real 2-dimensional plane in which $\text{Poly}(d^2)$ intersects the family $\{P_{\bar{c},c}\}_{c \in \mathbb{C}}$ by L . Since $P_{\bar{c}_0,c_0}$ is a post-critically finite hyperbolic polynomial (of degree d^2) with rational lamination λ_0 , we are now in the setting of renormalization and straightening maps defined over reduced mapping schemes. We refer the readers to [IK12, §1] for these general notions.

The combinatorial renormalization locus $\mathcal{C}(\lambda_0) := \{g \in \text{Poly}(d^2) : \lambda(g) \supset \lambda_0\}$ and the renormalization locus $\mathcal{R}(\lambda_0) = \{g \in \mathcal{C}(\lambda_0) : g \text{ is } \lambda_0\text{-renormalizable}\}$ satisfy $\mathcal{C}(c_0) = \mathcal{C}(\lambda_0) \cap L$ and $\mathcal{R}(c_0) = \mathcal{R}(\lambda_0) \cap L$. There is a straightening map $\chi_{\lambda_0} : \mathcal{R}(\lambda_0) \rightarrow \mathcal{C}(T(\lambda_0))$, where $\mathcal{C}(T(\lambda_0))$ is the fiber-wise connectedness locus of the family of monic centered polynomial maps over the reduced mapping scheme $T(\lambda_0)$ of λ_0 . Following [Ino14], we will now describe the set $\mathcal{C}(T(\lambda_0))$.

When k is even, $P_{\bar{c}_0,c_0}$ has two disjoint periodic cycles each containing a single critical point of multiplicity d (disjoint mapping scheme). This gives rise to two independent holomorphic unicritical-like maps (of degree d), and hence $\mathcal{C}(T(\lambda_0)) = \mathcal{M}_d \times \mathcal{M}_d$, which is the fiber-wise connectedness locus of the family:

$$\begin{aligned} \{P : \{0, 1\} \times \mathbb{C} \curvearrowright; P(k, z) = (k, p_{a_k}(z)), p_{a_k}(z) = z^d + a_k, a_k \in \mathbb{C}\} \\ = \{(p_{a_0}, p_{a_1}) : a_0, a_1 \in \mathbb{C}\} \cong \mathbb{C}^2. \end{aligned}$$

Now let $c \in \mathcal{R}(c_0)$. Every c_0 -renormalization $f_c^{\circ k} : U'_c \rightarrow U_c$ splits into two (holomorphic) unicritical-like maps (of degree d) $P_{\bar{c},c}^{\circ k/2} : U'_c \rightarrow U_c$ and

$P_{\bar{c},c}^{\circ k/2} : f_c(U'_c) \rightarrow f_c(U_c)$ (after shrinking U'_c and U_c if necessary). Moreover, the former (holomorphic) unicritical-like restriction is antiholomorphically conjugate to the latter one by f_c near the filled-in Julia sets. Therefore, as λ_0 -renormalization for $P_{\bar{c},c}$, we have two (holomorphic) unicritical-like maps of degree d which are antiholomorphically equivalent, and hence, the straightening of those are $p_{c'}$ and $p_{\bar{c}'}$ (recall that $p_c(z) = z^d + c$). Therefore, for any $c \in \mathcal{R}(c_0)$, $\chi_{\lambda_0}(P_{\bar{c},c}) = (p_{c'}, p_{\bar{c}'})$, for a unique $c' \in \mathcal{M}_d$ (by the condition of having a connected filled-in Julia set). On the other hand, the c_0 -renormalization $f_c^{\circ k} : U'_c \rightarrow U_c$ is holomorphic and unicritical-like of degree d . By the definition of straightening, $\chi_{c_0}(f_c) = p_{c'}$.

Now let k be odd. Then both the periodic critical points (of multiplicity d) of $P_{\bar{c}_0,c_0}$ lie on the same cycle (bitransitive mapping scheme). Therefore, $\mathcal{C}(T(\lambda_0))$ is the fiber-wise connectedness locus of the family:

$$\begin{aligned} \text{Poly}(d \times d) &= \{P : \{0, 1\} \times \mathbb{C} \ni P(k, z) = (1 - k, p_{a_k}(z)), \\ &\quad p_{a_k}(z) = z^d + a_k, a_k \in \mathbb{C}\} \\ &= \{(p_{a_0}, p_{a_1}) : a_0, a_1 \in \mathbb{C}\} \cong \mathbb{C}^2. \end{aligned}$$

Identifying any $P \in \text{Poly}(d \times d)$ with the composition $p_{a_1} \circ p_{a_0}$, we can view $\mathcal{C}(T(\lambda_0))$ as the connectedness locus of the family $\{(z^d + a)^d + b\}_{a,b \in \mathbb{C}}$.

Now let $c \in \mathcal{R}(c_0)$ and $\chi_{\lambda_0}(P_{\bar{c},c}) = (p_{a_0}, p_{a_1})$. The quartic-like map $P_{\bar{c},c}^{\circ k} : U'_c \rightarrow P_{\bar{c},c}^{\circ k}(U'_c) = f_c^{\circ k}(U_c)$ can be written as the composition of the two unicritical-like maps $Q_1 : P_{\bar{c},c}^{\circ \frac{k-1}{2}} : U'_c \rightarrow f_c^{\circ(k-1)}(U'_c)$ and $Q_2 : P_{\bar{c},c}^{\circ \frac{k+1}{2}} : f_c^{\circ(k-1)}(U'_c) \rightarrow f_c^{\circ k}(U_c)$, each of degree d . Note that $Q_2 \circ Q_1 = P_{\bar{c},c}^{\circ k} : U'_c \rightarrow f_c^{\circ k}(U_c)$ and $Q_1 \circ Q_2 = P_{\bar{c},c}^{\circ k} : f_c^{\circ(k-1)}(U'_c) \rightarrow f_c^{\circ(2k-1)}(U_c)$ are antiholomorphically conjugate by $f_c^{\circ(k-1)}$ near their filled-in Julia sets. Therefore, their straightening are antiholomorphically conjugate; i.e. $(p_{\bar{a}_1}, p_{\bar{a}_0}) = (p_{a_0}, p_{a_1})$. Therefore, $\chi_{\lambda_0}(P_{\bar{c},c}) = (p_{a_0}, p_{\bar{a}_0})$. Using the identification of $\text{Poly}(d \times d)$ with maps of the form $(z^d + a)^d + b$, we obtain that $\chi_{\lambda_0}(P_{\bar{c},c}) = p_{a_0} \circ p_{\bar{a}_0} = f_{a_0}^{\circ 2}$, for a unique $a_0 \in \mathcal{M}_d^*$, once we have fixed an external marking for our (anti-)polynomial-like maps (by the condition of having a connected filled-in Julia set). On the other hand, the c_0 -renormalization $f_c^{\circ k} : U'_c \rightarrow U_c$ is antiholomorphic and unicritical-like of degree d . By the definition of straightening, $\chi_{c_0}(f_c) = f_{a_0}$.

The above discussion (along with our chosen identifications) shows that the maps χ_{c_0} and χ_{λ_0} are essentially the same on $\mathcal{R}(c_0)$.

Define

$$\mathcal{M}(c_0) = \begin{cases} \mathcal{M}_d & \text{if } k \text{ is even,} \\ \mathcal{M}_d^* & \text{if } k \text{ is odd.} \end{cases}$$

Definition 7.3.2 (Straightening Map). *We call the map $\chi_{c_0} : \mathcal{R}(c_0) \rightarrow \mathcal{M}(c_0)$ as above the straightening map for c_0 .*

We call $\mathcal{C}(c_0)$ a *baby multibrot-like set* when the renormalization period is even. Otherwise, we call it a *baby multicorn-like set*.

By the rotational symmetry of the multicorns, if the period k is odd, then $\omega\chi_{c_0}, \omega^2\chi_{c_0}, \dots, \omega^d\chi_{c_0}$ are also straightening maps (with different internal/external markings), where $\omega = \exp(\frac{2\pi i}{d+1})$. In the sequel, we will always choose and fix one of them.

Definition 7.3.3. *We call a center $c_0 \in \mathcal{M}_d^*$ primitive if the closures of Fatou components of f_{c_0} are mutually disjoint.*

With these preparations, we are now ready to state the main results from [IK12] applied to our setting. Strictly speaking, these theorems hold for the map χ_{λ_0} , but we can apply them to the map χ_{c_0} since these two maps, suitably interpreted, agree on $\mathcal{R}(c_0)$.

Theorem 7.3.4 (Injectivity). *The straightening map $\chi_{c_0} : \mathcal{R}(c_0) \rightarrow \mathcal{M}(c_0)$ is injective.*

Theorem 7.3.5 (Onto Hyperbolicity). *The image $\chi_{c_0}(\mathcal{R}(c_0))$ of the straightening map contains all the hyperbolic components of $\mathcal{M}(c_0)$.*

Theorem 7.3.6 (Compactness). *If c_0 is primitive, then $\mathcal{C}(c_0) = \mathcal{R}(c_0)$, and it is compact.*

These general theorems also imply that in the even period case, the straightening map from a baby multibrot-like set to the original multibrot set is a homeomorphism (at least in good cases). See [Ino14, Appendix A] for a proof.

Theorem 7.3.7. *If c_0 is primitive, and the renormalization period is even, then the corresponding straightening map $\chi_{c_0} : \mathcal{R}(c_0) \rightarrow \mathcal{M}_d$ is a homeomorphism.*

7.4 Discontinuity of The Straightening Map

In this section, we will continue our study of the straightening map as developed in the previous section, and will prove our main theorem on the discontinuity of the straightening map in the odd period case.

It follows from an argument similar to Lemma 3.3.2 that the centers of odd period hyperbolic components of $\mathcal{M}_d^*$ are primitive. Using Theorem 7.3.6, we conclude that:

Corollary 7.4.1. *If the renormalization period is odd, then $\mathcal{C}(c_0) = \mathcal{R}(c_0)$, and it is compact.*

Before giving the proof of the Theorem 1.2.8, we need to show the existence of hyperbolic components in the real part of $\mathcal{M}_d^*$.

Lemma 7.4.2. *If d is even, then at least one hyperbolic component of period 3 of $\mathcal{M}_d^*$ intersects $\mathbb{R} \cap \mathcal{M}_d^*$.*

Proof. Let S_d be the set of all hyperbolic components of period 3 of $\mathcal{M}_d^*$. By 1.2.3, $|S_d| = d^2 - 1$. Note that $\mathcal{M}_d^*$ has a complex conjugation symmetry; i.e. complex conjugation induces an involutive bijection on the set S_d . When d is even, $|S_d|$ is an odd integer, and this implies that there must be some element H_d^* in S_d fixed by complex conjugation. Clearly, H_d^* intersects the real line. Moreover, the center of H_d^* and the critical Ecalle height 0 parameter on the root arc of ∂H_d^* are real, and a piece of $\mathbb{R} \cap \mathcal{M}_d^*$ converges to this critical Ecalle height 0 parameter from the exterior of $\overline{H_d^*}$. $\square$

Proof of Theorem 1.2.8. Let d be even, and c_0 be the center of a hyperbolic component of odd period k ($k \neq 1$) of $\mathcal{M}_d^*$. We will assume that the map $\chi_{c_0} : \mathcal{R}(c_0) \rightarrow \mathcal{M}_d^*$ is continuous, and will arrive at a contradiction.

By Corollary 7.4.1 and Theorem 7.3.4, $\mathcal{R}(c_0)$ is compact, and the map χ_{c_0} is injective. Since an injective continuous map from a compact topological space onto a Hausdorff topological space is a homeomorphism, it follows that χ_{c_0} is a homeomorphism from $\mathcal{R}(c_0)$ onto its range (we do not claim that $\chi_{c_0}(\mathcal{R}(c_0)) = \mathcal{M}_d^*$). In particular, $\chi_{c_0}(\mathcal{R}(c_0))$ is closed. Since hyperbolic real polynomials are dense in the family of real polynomials [KSvS07], it follows from Theorem 7.3.5 and the $d + 1$ -fold rotational symmetry of the multicorns that $(\mathbb{R} \cup \omega\mathbb{R} \cdots \cup \omega^d\mathbb{R}) \cap \mathcal{M}_d^* \subset \chi_{c_0}(\mathcal{R}(c_0))$. Moreover, by Theorem 7.3.5, $H_d^* \cup \omega H_d^* \cup \cdots \cup \omega^d H_d^* \subset \chi_{c_0}(\mathcal{R}(c_0))$.

Since c_0 is not of period 1, there exists $i \in \{0, 1, \dots, d\}$ such that $H' := \chi_{c_0}^{-1}(\omega^i H_d^*)$ does not intersect the real line or its ω -rotates (but is contained in $\mathcal{R}(c_0)$). Recall that there exists a piece γ of $\omega^i \mathbb{R} \cap \mathcal{M}_d^*$ that lies outside of $\overline{\omega^i H_d^*}$, and lands at the critical Ecalle height 0 parameter on the root parabolic arc of $\partial(\omega^i H_d^*)$. By our assumption, χ_{c_0} is a homeomorphism; and hence the curve $\chi_{c_0}^{-1}(\gamma)$ lies in the exterior of $\overline{H'}$, and lands at the critical Ecalle height 0 parameter on the root arc of $\partial H'$ (critical Ecalle heights are preserved by hybrid equivalences). Since H' does not intersect the real line or its ω -rotates, this contradicts Theorem 1.2.9.

The hyperbolic component of period 3 (intersecting the real line) does not play any special role in the above proof; in fact, there are infinitely many odd period hyperbolic components of $\mathcal{M}_d^*$ that intersect the real line. Our argument applies verbatim to any of these hyperbolic components, which proves that the straightening map is indeed discontinuous at infinitely many parameters. $\square$

Remark 7.4.3. *Observe that our proof demonstrates the discontinuity of χ_{c_0} at an explicit parameter; for instance, at the critical Ecalle height 0 parameter on the root parabolic arc of $\partial H'$.*

7.5 Are All Baby Multicorns Dynamically Distinct?

We have proved in the previous section that for multicorns of even degree, the straightening map from a baby multicorn-like set based at an odd period (different from 1) hyperbolic component to the multicorn is discontinuous at infinitely many critical Ecalle height 0 parameters. However, we conjecture that there is a stronger form of discontinuity; i.e any two baby multicorn-like sets (for multicorns of any degree) are dynamically different.

For $i = 1, 2$, let H_i be a hyperbolic component of odd period k_i of $\mathcal{M}_d^*$, $\mathcal{C}_i$ be the root parabolic arc on ∂H_i , and $c_i : \mathbb{R} \rightarrow \mathcal{C}_i$ be the critical Ecalle height parametrization of $\mathcal{C}_i$. By Theorem 1.1.20, each end of $\mathcal{C}_i$ intersects the boundary of a hyperbolic component of period $2k_i$ at the set of points where the residue fixed-point index of the unique parabolic cycle is at least 1, except possibly at (necessarily isolated) points where the index has an isolated local maximum with value 1. Let $c_1(h_1)$ and $c_1(h_2)$ be the parameters on $\mathcal{C}_1$ (respectively $c_2(h_3)$ and $c_2(h_4)$ be the parameters on $\mathcal{C}_2$) at which bifurcations from H_1 (respectively from H_2) to these hyperbolic components of period $2k_1$ (respectively $2k_2$) starts (compare 7.4).

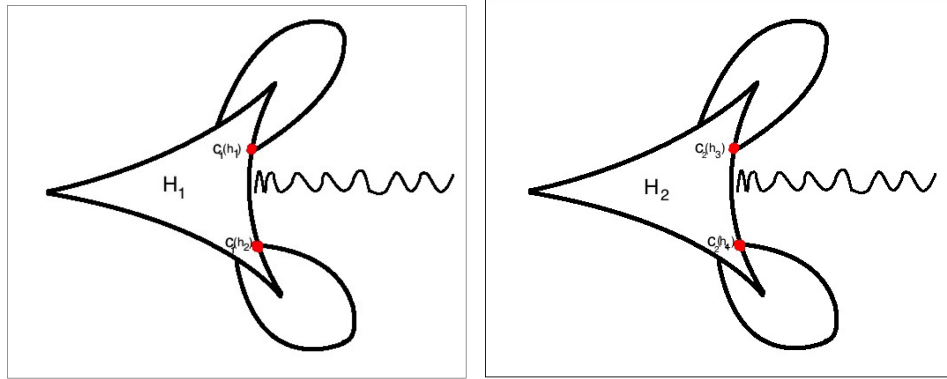


Figure 7.4: Left: Period doubling bifurcation from H_1 starting at parameters $c_1(h_1)$ and $c_1(h_2)$. Right: Period doubling bifurcation from H_2 starting at parameters $c_2(h_3)$ and $c_2(h_4)$.

Recall that critical Ecalle heights are preserved by hybrid equivalences, and hence by the straightening map. Furthermore, since the straightening map respects the bifurcation structure, a necessary condition for the two multicorn-like sets based at H_1 and H_2 (respectively) to be dynamically equivalent is that $\{h_1, h_2\} = \{h_3, h_4\}$. However, these parameters are characterized by the condition of having residue fixed-point index (of their unique parabolic cycle) $+1$, and this being an ‘external conformal information’ for the polynomial-like restrictions (the hybrid class of a polynomial-like map does not determine the

fixed point index), has no reason to be preserved by hybrid equivalences. We therefore make the following conjecture:

Conjecture 7.5.1 (Baby Multicorns Are Dynamically Distinct). *If $\iota(H_1) \neq H_2$ and $\omega^i H_1 \neq H_2$ for $i = 1, 2, \dots, d+1$, then $\{h_1, h_2\} \neq \{h_3, h_4\}$.*

Appendix A

Tricorns in Real Cubics

In this section, we will discuss the topological properties of the umbilical cords for the tricorn-like sets in the family of real cubic polynomials. The strategy (and the techniques) would be similar to the one used in the proof of Theorem 1.2.8, so we will omit some of the details. Let us begin by elaborating the set-up of tricorn-like sets in the family of real cubic polynomials. We will work with the family:

$$\mathcal{G} = \{g_{a,b}(z) = -z^3 - 3a^2z + b, \ a \geq 0, b \in \mathbb{R}\}.$$

$g_{a,b}$ commutes with complex conjugation ι ; i.e. $g_{a,b}$ has a reflection symmetry with respect to the real line. Observe that $g_{a,b}$ is conjugate to the monic centered polynomial $h_{a,b}(z) = z^3 - 3a^2z + bi$ by the affine map $z \mapsto iz$, and $h_{a,b}$ has a reflection symmetry with respect to the imaginary axis (i.e. $h_{a,b}(-\iota(z)) = -\iota(h_{a,b}(z))$, where $\iota(z) = \bar{z}$). We will, however, work with the real form $g_{a,b}$, and will normalize the Böttcher coordinate $\varphi_{a,b}$ of $g_{a,b}$ (at ∞) such that $\varphi_{a,b}(z)/z \rightarrow -i$ as $z \rightarrow \infty$. Also note that $-g_{a,b}(-z) = g_{a,-b}(z)$. Nonetheless, to define the straightening map consistently, we need to distinguish $g_{a,b}$ and $g_{a,-b}$ as they have different rational laminations (with respect to our normalized Böttcher coordinates). We will denote the connectedness locus of $\mathcal{G}$ by $\mathcal{C}(\mathcal{G})$.

The two critical points of $\pm ai$ (as well as their forward orbits) of $g_{a,b}$ are symmetric with respect to the real axis (since $g_{a,b}$ commutes with complex conjugation). Let (a_0, b_0) be such that there exists an integer $n > 1$ such that $g_{a_0, b_0}^{\circ n}(ia_0) = -ia_0$ and $g_{a_0, b_0}^{\circ n}(-ia_0) = ia_0$. If U is a sufficiently small neighborhood of the closure of the Fatou component containing ia_0 , then we have, $\iota(\bar{U}) \subset g_{a_0, b_0}^{\circ n}(U)$ (where $\bar{U}$ is the topological closure of U , and ι is the complex conjugation map), i.e. $\bar{U} \subset (\iota \circ g_{a_0, b_0}^{\circ n})(U)$ (compare Figure 1.3). Since $\iota \circ g_{a_0, b_0}^{\circ n}$ is an antiholomorphic map of degree 2, we have an anti-polynomial-like map of degree 2 (with a connected filled-in Julia set) defined on U . The straightening Theorem 1.1.25 now yields a quadratic anti-polynomial (with a connected filled-in Julia set) that is hybrid equivalent to $(\iota \circ g_{a_0, b_0}^{\circ n})|_U$. One can continue

to perform this renormalization procedure as the real cubic polynomial $g_{a,b}$ moves in the parameter space, and this defines a map from a suitable region in the parameter plane of real cubic polynomials to the tricorn.

More precisely, let $\lambda_{a,b}$ be the rational lamination of $g_{a,b}$. Define the combinatorial renormalization locus to be

$$\mathcal{C}(a_0, b_0) = \{(a, b) \in \mathbb{R}^2 : a > 0, \lambda_{a,b} \supset \lambda_{a_0, b_0}\}$$

and the renormalization locus to be

$$\mathcal{R}(a_0, b_0) = \{(a, b) \in \mathcal{C}(a_0, b_0) : \exists U', U \text{ such that } \iota \circ g_{a,b}^{\circ n} : U' \rightarrow U \text{ is anti-polynomial-like of degree 2 with a connected filled-in Julia set}\}.$$

Using Theorem 1.1.25, for each $(a, b) \in \mathcal{R}(a_0, b_0)$, we can straighten $\iota \circ g_{a,b}^{\circ n} : U' \rightarrow U$ to obtain a quadratic anti-polynomial f_c . This defines the straightening map

$$\begin{aligned} \chi_{a_0, b_0} : \mathcal{R}(a_0, b_0) &\rightarrow \mathcal{M}_2^* \\ (a, b) &\mapsto c. \end{aligned}$$

The condition $n > 1$ implies that g_{a_0, b_0} is primitive. The proof of the fact that our definition of χ_{a_0, b_0} agrees with the general definition of straightening maps [IK12] goes as in Section 7.3. Therefore, the analogues of Theorem 7.3.4, Theorem 7.3.5 and Theorem 7.3.6 hold in this setting as well.

Let H_1, H_2, H_3 be the hyperbolic components of period 3 of $\mathcal{M}_2^*$ (by Theorem 3.6, there are only 3 of them). Since $\chi_{a_0, b_0}(\mathcal{R}(a_0, b_0))$ contains the hyperbolic components of $\mathcal{M}_2^*$, there exists $i \in \{1, 2, 3\}$ such that $H' := \chi_{a_0, b_0}^{-1}(H_i)$ does not intersect the real line (but is contained in $\mathcal{R}(a_0, b_0)$). Since for any $(a, b) \in \mathcal{R}(a_0, b_0)$, $\iota \circ g_{a,b}^{\circ n}$ has an anti-polynomial-like restriction of degree 2, one can argue as in the unicritical antiholomorphic case that $\partial H'$ consists of 3 parabolic cusps and 3 parabolic arcs which are parametrized by the critical Ecalle height. Let $(\tilde{a}, \tilde{b})$ be the critical Ecalle height 0 parameter on the root arc (such that the unique parabolic cycle disconnects the Julia set) of $\partial H'$. Let U be the unique Fatou component of $g_{\tilde{a}, \tilde{b}}$ containing the critical point ai and $\tilde{z}$ be the unique parabolic periodic point of $g_{\tilde{a}, \tilde{b}}$ on ∂U . Since $g_{\tilde{a}, \tilde{b}}$ commutes with ι , $\iota(U)$ is the unique Fatou component of $g_{\tilde{a}, \tilde{b}}$ containing the critical point $-ai$ and $\tilde{z}^*$ is the unique parabolic periodic point of $g_{\tilde{a}, \tilde{b}}$ on $\partial \iota(U)$. We will show that if the ‘umbilical cord’ of H' lands, then the two polynomial-like restrictions of $g_{\tilde{a}, \tilde{b}}^{\circ 2n}$ in some neighborhoods of $\bar{U}$ and $\overline{\iota(U)}$ (respectively) are conformally conjugate.

Lemma A.0.1. *If there exists a path $p : [0, \delta] \rightarrow \mathbb{R}^2$ such that $p(0) = (\tilde{a}, \tilde{b})$ and $p((0, \delta]) \subset \mathcal{C}(\mathcal{G}) \setminus \bar{H}'$, then the two polynomial-like restrictions of $g_{\tilde{a}, \tilde{b}}^{\circ 2n}$ in some neighborhoods of $\bar{U}$ and $\overline{\iota(U)}$ respectively are conformally conjugate.*

Proof. The proof is essentially the same as for the maps f_c , so we only give a sketch.

Applying the parabolic implosion techniques (Lemma 5.1.1), one shows that the existence of such a path p would imply that the repelling equator at $\tilde{z}$ is contained in a loose parabolic tree of $g_{\tilde{a},\tilde{b}}$. Note that the proof of this fact in Lemma 7.1.2 made use of the parameter rays of the multicorns. However, one can circumvent that by the following argument. If the repelling equator at $\tilde{z}$ is not contained in a loose parabolic tree of $g_{\tilde{a},\tilde{b}}$, then there would exist dynamical rays (in the dynamical plane of $g_{\tilde{a},\tilde{b}}$) traversing an interval of outgoing Ecalle heights $[-x, x]$ with $x > 0$. This would remain true after perturbation. For $s > 0$, the critical orbits of $g_{p(s)}$ “transit” from the incoming Ecalle cylinder to the outgoing cylinder (the two critical orbits are related by the conjugacy ι); as $s \downarrow 0$, the image of the critical orbits in the outgoing Ecalle cylinder has (outgoing) Ecalle height tending to 0, while the phase tends to $-\infty$ (by Lemma 5.1.1). Therefore, there exists $s \in (0, \delta)$ arbitrarily close to 0 for which the critical orbit(s), projected into the incoming cylinder, and sent by the transit map to the outgoing cylinder, land(s) on the projection of some dynamical ray that crosses the equator. But in the dynamics of $g_{p(s)}$, this means that the critical orbits lie in the basin of infinity, i.e. such a parameter $p(s)$ lies outside $\mathcal{C}(\mathcal{G})$. This contradicts our assumption that $p((0, \delta]) \subset \mathcal{C}(\mathcal{G}) \setminus \overline{H'}$.

If the repelling equator at $\tilde{z}$ is contained in a loose parabolic tree of $g_{\tilde{a},\tilde{b}}$, then one can argue as in Lemma 7.1.3 to conclude that there exists a real-analytic arc Γ , which is invariant under $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ and passes through $\tilde{z}$. This implies that $g_{\tilde{a},\tilde{b}}^{\circ 2n}|_{B(\tilde{z},\varepsilon)}$ (for ε sufficiently small) is conformally conjugate to a real parabolic germ.

One can carry out the same argument with the parabolic periodic point $\tilde{z}^*$ (as in Lemma 7.1.3) to conclude that $g_{\tilde{a},\tilde{b}}^{\circ 2n}|_{B(\tilde{z}^*,\varepsilon)}$ is conformally conjugate to the same real parabolic germ as the one constructed in the previous paragraph. Thus, we have a local biholomorphism η between the parabolic germs $g_{\tilde{a},\tilde{b}}^{\circ 2n}|_{B(\tilde{z},\varepsilon)}$ and $g_{\tilde{a},\tilde{b}}^{\circ 2n}|_{B(\tilde{z}^*,\varepsilon)}$ such that η maps $g_{\tilde{a},\tilde{b}}^{\circ 2rn}(ai)$ to $g_{\tilde{a},\tilde{b}}^{\circ 2rn}(-ai)$, for r large enough so that $g_{\tilde{a},\tilde{b}}^{\circ 2rn}(ai)$ lies in the domain of definition of η .

Observe that $g_{\tilde{a},\tilde{b}}^{\circ 2n}|_U$ and $g_{\tilde{a},\tilde{b}}^{\circ 2n}|_{\iota(U)}$ are conformally conjugate (by the condition of having critical Ecalle height 0). Hence, we can extend η to a conformal conjugacy between the polynomial-like restrictions of $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ in some neighborhoods of $\overline{U}$ and $\overline{\iota(U)}$ respectively (arguing as in Lemma 7.2.2 and Lemma 7.2.3). $\square$

Remark A.0.2. *We would like to emphasize, albeit at the risk of being pedantic, that the germ conjugacy η extends to the closure of the basins only because it respects the dynamics on the critical orbits.*

$g_{\tilde{a},\tilde{b}}^{\circ 2n}$ has three critical points $u, v, c = ai$ in the Fatou component U , such that $g_{\tilde{a},\tilde{b}}^{\circ 2n}(u) = g_{\tilde{a},\tilde{b}}^{\circ 2n}(v) = g_{\tilde{a},\tilde{b}}^{\circ n}(-ai) = g_{\tilde{a},\tilde{b}}^{\circ n}(c^*)$. Therefore $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ has two

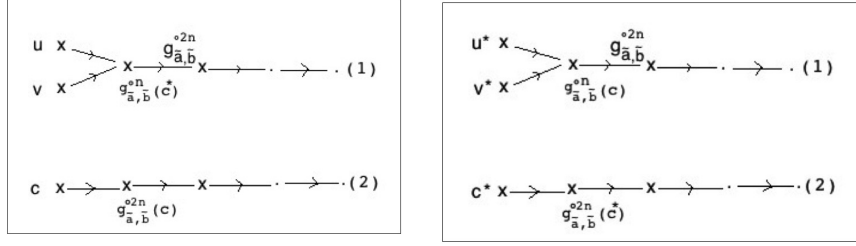


Figure A.1: Left: The dynamics on the two dynamically marked critical orbits of $g_{a,b}^{o2n}|_U$. Right: The dynamics on the two dynamically marked critical orbits of $g_{\tilde{a},\tilde{b}}^{o2n}|_{\iota(U)}$. Observe that $g_{\tilde{a},\tilde{b}}^{on}$ swaps the two dynamically marked critical orbits.

infinite critical orbits in U , and these two critical orbits are dynamically different. By our construction, η maps (the tail of) each of the two (dynamically distinct) critical orbits of $g_{a,b}^{o2n}|_U$ to the (tail of the) corresponding critical orbit of $g_{\tilde{a},\tilde{b}}^{o2n}|_{\iota(U)}$. In fact, the parabolic germs $g_{a,b}^{o2n}|_{B(\tilde{z},\varepsilon)}$ and $g_{\tilde{a},\tilde{b}}^{o2n}|_{B(\tilde{z}^,\varepsilon)}$ are always conformally conjugate by $g_{a,b}^{on}$; but this local conjugacy exchanges the two dynamically marked post-critical orbits (compare Figure A.1), which have different topological dynamics, and hence this local conjugacy has no chance of being extended to the entire parabolic basin.*

Theorem A.0.3 (Umbilical Cord Wiggling in Real Cubics). *There does not exist a path $p : [0, \delta] \rightarrow \mathbb{R}^2$ such that $p(0) = (\tilde{a}, \tilde{b})$ and $p((0, \delta]) \subset \mathcal{C}(\mathcal{G}) \setminus \overline{H'}$.*

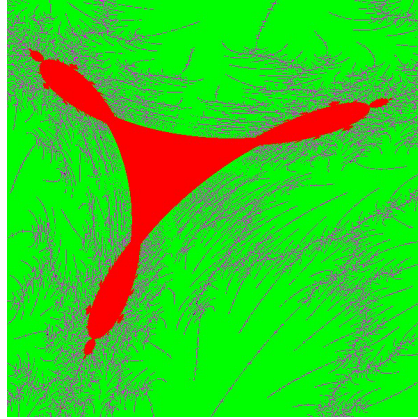


Figure A.2: Wiggling of an umbilical cord for a tricorn-like set in the real cubic locus.

Proof. We have already showed that the existence of the required path implies that the polynomial-like restrictions $g_{\tilde{a},\tilde{b}}^{\circ 2n} : U' \rightarrow g_{\tilde{a},\tilde{b}}^{\circ 2n}(U')$ (where U' is a neighborhood of $\overline{U}$) and $g_{\tilde{a},\tilde{b}}^{\circ 2n} : \iota(U') \rightarrow g_{\tilde{a},\tilde{b}}^{\circ 2n}(\iota(U'))$ (where $\iota(U')$ is a neighborhood of $\iota(\overline{U})$) are conformally conjugate. Applying Theorem [Ino11, Theorem 1] to this situation and arguing as in Lemma 7.2.4, we obtain the existence of polynomials p, p_1 and p_2 such that

$$g_{\tilde{a},\tilde{b}}^{\circ 2n} \circ p_1 = p_1 \circ p, \quad g_{\tilde{a},\tilde{b}}^{\circ 2n} \circ p_2 = p_2 \circ p, \quad \text{and } \deg p_1 = \deg p_2.$$

Moreover, by Theorem [Ino11, Theorem 1], p has a polynomial-like restriction $p : V \rightarrow p(V)$ which is conformally conjugate to $g_{\tilde{a},\tilde{b}}^{\circ 2n} : U' \rightarrow g_{\tilde{a},\tilde{b}}^{\circ 2n}(U')$ by p_1 and to $g_{\tilde{a},\tilde{b}}^{\circ 2n} : \iota(U') \rightarrow g_{\tilde{a},\tilde{b}}^{\circ 2n}(\iota(U'))$ by p_2 . We now consider two cases.

Case 1 ($\deg(p_1) = \deg(p_2) = 1$). Set $p_3 := p_1 \circ p_2^{-1}$. Then p_3 is an affine map commuting with $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ and conjugating the two polynomial-like restrictions of $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ under consideration. Clearly, $p_3 \neq \text{id}$. An easy computation (using the fact that $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ is a centered real polynomial) now shows that $p_3(z) = -z$, and hence $\tilde{b} = 0$.

Case 2 ($\deg(p_1) = \deg(p_2) = k > 1$). The arguments employed in this case are morally similar to the last step in the proof of Theorem 1.2.9. We will first prove by contradiction that $\gcd(\deg g_{\tilde{a},\tilde{b}}^{\circ 2n}, \deg p_1) > 1$. To do this, let $\gcd(\deg g_{\tilde{a},\tilde{b}}^{\circ 2n}, \deg p_1) = 1$. Now we can apply Theorem 7.2.6 to our situation. Since $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ is parabolic, it is neither a power map, nor a Chebyshev polynomial. Hence, there exists some non-constant polynomial P such that $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ is affinely conjugate to the polynomial $h(z) := z^r(P(z))^k$ and $p_1(z) = z^k$ (up to affine conjugacy). If $r \geq 2$, then $h(z)$ has a super-attracting fixed point at 0. But $g_{\tilde{a},\tilde{b}}^{\circ 2n}$, which is affinely conjugate to $h(z)$, has no super-attracting fixed point. Hence, $r = 0$ or 1. By degree consideration, we have $3^{2n} = r + ks$, where $\deg P = s$. The assumption $\gcd(\deg g_{\tilde{a},\tilde{b}}^{\circ 2n}, \deg p_1) = \gcd(3^{2n}, k) = 1$ implies that $r = 1$, i.e. $h(z) = z(P(z))^k$. Now the fixed point 0 for h satisfies $h^{-1}(0) = \{0\} \cup P^{-1}(0)$, and any point in $P^{-1}(0)$ has a local mapping degree k under h . The same must hold for the affinely conjugate polynomial $g_{\tilde{a},\tilde{b}}^{\circ 2n}$: there exists a fixed point (say x) for $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ such that any point in $(g_{\tilde{a},\tilde{b}}^{\circ 2n})^{-1}(x)$ has mapping degree k except for x ; in particular, all points in $(g_{\tilde{a},\tilde{b}}^{\circ 2n})^{-1}(x) \setminus \{x\}$ are critical points for $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ (since $k > 1$). But this implies that $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ has a finite critical orbit, which is a contradiction to the fact that all critical orbits of $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ non-trivially converge to parabolic fixed points.

Now applying Engstrom's theorem [Eng41], we obtain the existence of polynomials (of degree at least two) α, β such that

$$g_{\tilde{a},\tilde{b}}^{\circ 2n} = \alpha \circ \beta, \quad p = \beta \circ \alpha.$$

Since $g_{\tilde{a},\tilde{b}}$ is a prime polynomial under composition (since its degree is a prime number), it follows that $p = g_{\tilde{a},\tilde{b}}^{\circ 2n}$. Therefore, p_1 commutes with $g_{\tilde{a},\tilde{b}}^{\circ 2n}$. As $g_{\tilde{a},\tilde{b}}^{\circ 2n}$ is neither a power map nor a Chebychev polynomial, $p_1 = g_{\tilde{a},\tilde{b}}^{\circ k_1}$, for some $k_1 \in \mathbb{N}$ (up to affine conjugacy). The same is true for p_2 as well; i.e. $p_2 = g_{\tilde{a},\tilde{b}}^{\circ k_2}$ (up to affine conjugacy).

Therefore, there is a polynomial-like restriction of $p = g_{\tilde{a},\tilde{b}}^{\circ 2n} : V \rightarrow g_{\tilde{a},\tilde{b}}^{\circ 2n}(V)$, which is conformally conjugate to $g_{\tilde{a},\tilde{b}}^{\circ 2n} : U' \rightarrow g_{\tilde{a},\tilde{b}}^{\circ 2n}(U')$ by $p_1 = g_{\tilde{a},\tilde{b}}^{\circ k_1}$ and to $g_{\tilde{a},\tilde{b}}^{\circ 2n} : \iota(U') \rightarrow g_{\tilde{a},\tilde{b}}^{\circ 2n}(\iota(U'))$ by $p_2 = g_{\tilde{a},\tilde{b}}^{\circ k_2}$. But the dynamical configuration implies that this is impossible (since there is only one parabolic cycle and the unique cycle of immediate parabolic basins contains two critical points of $g_{\tilde{a},\tilde{b}}$, either p_1 or p_2 must have a critical point).

Therefore, we have showed that the existence of such a path p would imply that $\tilde{b} = 0$. But this contradicts our assumption that H' does not intersect the real line. This completes the proof of the theorem. $\square$

Using Theorem A.0.3, we can now proceed as in Theorem 1.2.8 to prove that the straightening map $\chi_{a_0,b_0} : \mathcal{R}(a_0,b_0) \rightarrow \mathcal{M}_2^*$ is discontinuous.

Theorem A.0.4 (Discontinuity of Straightening II). *The straightening map $\chi_{a_0,b_0} : \mathcal{R}(a_0,b_0) \rightarrow \mathcal{M}_2^*$ is discontinuous (at infinitely many explicit parameters).*

Proof. We will stick to the terminologies used throughout this appendix. We will assume that the map $\chi_{a_0,b_0} : \mathcal{R}(a_0,b_0) \rightarrow \mathcal{M}_2^*$ is continuous, and will arrive at a contradiction.

We have observed that $\mathcal{R}(a_0,b_0)$ is compact, and the map χ_{a_0,b_0} is injective. Since an injective continuous map from a compact topological space onto a Hausdorff topological space is a homeomorphism, it follows that χ_{a_0,b_0} is a homeomorphism from $\mathcal{R}(a_0,b_0)$ onto its range (we do not claim that $\chi_{a_0,b_0}(\mathcal{R}(a_0,b_0)) = \mathcal{M}_2^*$). In particular, $\chi_{a_0,b_0}(\mathcal{R}(a_0,b_0))$ is closed. Since real hyperbolic quadratic polynomials are dense in $\mathbb{R} \cap \mathcal{M}_2^*$ [GS97, Lyu97] (the tricorn and the Mandelbrot set agree on the real line), it follows from Theorem 7.3.5 and the 3-fold rotational symmetry of the tricorn that $(\mathbb{R} \cup \omega\mathbb{R} \cup \omega^2\mathbb{R}) \cap \mathcal{M}_2^* \subset \chi_{a_0,b_0}(\mathcal{R}(a_0,b_0))$.

Recall that there exists a piece γ of $\omega^i\mathbb{R} \cap \mathcal{M}_2^*$ that lies outside of $\overline{H_i}$ (where $i \in \{1, 2, 3\}$ and H_1, H_2, H_3 are the hyperbolic components of period 3 of $\mathcal{M}_2^*$), and lands at the critical Ecalle height 0 parameter on the root parabolic arc of ∂H_i . By our assumption, χ_{a_0,b_0} is a homeomorphism; and hence the curve $\chi_{a_0,b_0}^{-1}(\gamma)$ lies in the exterior of $\overline{H'}$, and lands at the critical Ecalle height 0 parameter on the root arc of $\partial H'$ (critical Ecalle heights are preserved by hybrid equivalences). But this contradicts Theorem A.0.3. $\square$

Appendix B

Relation between Conformal Invariants

This section will contain a rough description of certain geometric features of attracting and parabolic basins with multiple critical points. We will assign some conformal conjugacy invariants to such basins, which in good circumstances, would determine the conformal class of the basin. Much of what follows is motivated by our study of unicritical anti-polynomials. The following result, which states that any two unicritical parabolic basins are conformally conjugate, is classical.

Lemma B.0.5 (Unisingular Parabolic Basins). *Let $p : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a rational map with a parabolic periodic point z , U be a simply connected immediate basin of attraction of z , and the first return map $p^{\circ n} : U \rightarrow U$ has a unique critical point of multiplicity d . Then $p^{\circ n} : U \rightarrow U$ is conformally conjugate to the Blaschke product $B_d : \mathbb{D} \rightarrow \mathbb{D}$, where $B_d(w) = \frac{(d+1)w^d + (d-1)}{(d+1) + (d-1)w^d}$.*

The situation becomes much more complicated if an immediate basin contains more than one critical points. For two such basins $p_1 : U_1 \rightarrow U_1$ and $p_2 : U_2 \rightarrow U_2$ to be conformally conjugate, several conditions need to be satisfied. Firstly, the topological dynamics along the critical orbits must be preserved, i.e the critical points of p_1 in U_1 must be in a $1 - 1$ correspondence with the critical points of p_2 in U_2 such that the correspondence preserves the multiplicities of the critical points. Furthermore, they must also have identical critical orbit relations (if any). The more serious obstruction, however, comes from geometric rigidity; i.e. the relative positions of the critical values of $p_i|_{U_i}$.

Let us explain this in an illustrative case. Consider two parabolic polynomials $f_{c_1}^{\circ 2}$ and $f_{c_2}^{\circ 2}$ (recall that $f_c(z) = \bar{z}^d + c$) with odd periodic parabolic cycles. Then their critical orbits have equivalent topological dynamics. But the characteristic Fatou components U_i of $f_{c_i}^{\circ 2}$ ($i = 1, 2$) have conformally conjugate dynamics if and only if the critical Ecalle height of c_1 and the critical Ecalle height of c_2 are equal. In fact, we have extended attracting Fatou coordinates

$\psi_{c_i}^{\text{att}} : U_i \rightarrow \mathbb{C}$, which are ramified covers ramified only over the pre-critical points in U_i . The condition of having equal critical Ecalle height implies that the corresponding pre-critical points have equal Fatou coordinates, and hence $(\psi_{c_1}^{\text{att}})^{-1} \circ \psi_{c_2}^{\text{att}}$ induces a conformal isomorphism from U_2 onto U_1 . Recall from Theorem 4.1.3 that the critical Ecalle height essentially measures the difference of the Fatou coordinates of the critical values of the first (holomorphic) return map of the characteristic Fatou component. Therefore this difference, which we call the *Fatou vector*, is a complete conformal conjugacy invariant of the characteristic Fatou component. More generally, we can define Fatou vectors for immediate basins containing two infinite critical orbits. A particular case is cubic polynomials with a unique parabolic cycle attracting both critical orbits.

In all these cases, one can also associate another conformal conjugacy invariant to the parabolic cycle, namely the residue fixed point index (see Section 1.1 for the definition). It is natural to ask how these two conformal invariants, namely the Fatou vector and the fixed point index, are related. This has parameter space implications; e.g. the Fatou vector is preserved by the straightening map, but the fixed point index is typically not, and hence an understanding of their relation will provide a better understanding of the straightening map. Here is the precise version of this question for the maps $P_{a,b}(z) = (z^d + a)^d + b$.

Recall from Section 4.1 that for a given parabolic arc $\mathcal{C}$ of period k of $\mathcal{M}_d^*$, there exists an injective holomorphic map

$$F : \{w = u + iv \in \mathbb{C} : |v| < \tfrac{1}{4}\} \rightarrow P_k, \\ w \mapsto (a(w), b(w))$$

such that for any $w \in \mathbb{R}$, $F(w) = (\overline{c(w)}, c(w))$ where $c : \mathbb{R} \rightarrow \mathcal{C}$ is the critical Ecalle height parametrization of the parabolic arc. Further, all the polynomials $P_{a(w),b(w)}$ have q.c.-conjugate (but not conformally conjugate) dynamics. The Fatou vector of $P_{a(w),b(w)}$ is $\frac{1}{2} - 2iw$.

Question B.0.6 (Relation between Fatou Vector and Fixed Point Index). *What can be said about the precise form of the holomorphic function*

$$\{w = u + iv \in \mathbb{C} : |v| < \tfrac{1}{4}\} \rightarrow \mathbb{C} \\ w \mapsto \iota(P_{a(w),b(w)}^{\circ k}, z_w),$$

where z_w is a parabolic periodic point of period k of $P_{a(w),b(w)}$?

There is an alternative formulation of the previous question involving the geometry of the immediate basin of the parabolic points. For any $w \in \{w = u + iv \in \mathbb{C} : |v| < \frac{1}{4}\}$, the first return map $P_{a(w),b(w)}^{\circ k}$ of the immediate parabolic basin is conformally conjugate to a unique Blaschke product B_w with a double parabolic fixed point. By [BE02, §3], $\iota(P_{a(w),b(w)}^{\circ k}, z_w)$ is related to the Grötzsch

defect of the compact set defined as the complement of the immediate parabolic basin in the repelling Ecalle cylinder and the residue fixed point index of B_w at its parabolic point. We denote this compact set by E_w and its Grötzsch defect by $\text{defect}(E_w)$. The quantity $\text{defect}(E_w)$ measures, for instance, the size of the domain of definition of the extended horn maps of $P_{a(w),b(w)}$ (see Appendix C for the definition of extended horn maps).

Question B.0.7 (Relation between Fatou Vector and Grötzsch defect). *What can be said about the precise form of the function*

$$\begin{aligned} \{w = u + iv \in \mathbb{C} : |v| < \tfrac{1}{4}\} &\rightarrow \mathbb{C} \\ w &\mapsto \text{defect}(E_w)? \end{aligned}$$

Let us now turn our attention to immediate basins of attracting cycles. Let H be a hyperbolic component of odd period k of $\mathcal{M}_d^*$. We want to associate a conformal invariant to $f_c^{\circ 2}$, for each $c \in H$. In this situation, there are two distinct critical orbits of $f_c^{\circ 2}$ converging to the unique attracting cycle. One can choose two representatives of these two critical orbits in a fundamental domain (say, in the critical value Fatou component), and consider their ratio in a holomorphic Koenig's coordinate. This ratio is well-defined as the choice of Koenig's coordinate does not affect it, and is a conformal conjugacy invariant of $f_c^{\circ 2}$. We will call it the *Koenig's ratio*, and will denote it by λ_c . By [NS03, §5], the map $c \mapsto \lambda_c$ is a $d + 1$ to 1 real-analytic branched covering from H to $\mathbb{D}$. This allows us to define internal rays of H as pre-images of radial lines (note that this would define $d + 1$ internal rays at a given angle).

Question B.0.8 (Limits of Internal Rays). *1. Does every internal ray of H land on ∂H ?*

2. if $\tilde{c} \in \partial H$ is an accumulation point of an internal ray at angle ϑ of H , can we relate ϑ with the critical Ecalle height/Fatou vector of $\tilde{c}$?

Appendix C

Recovering Cauliflowers from Parabolic Germs

Recall that the one of the key steps in the proof of Theorem 1.2.9 was to extend a carefully constructed local (germ) conjugacy to a semi-local (polynomial-like map) conjugacy, which allowed us to conclude that the corresponding polynomials are affinely conjugate. The extension of the germ conjugacy (see Lemma 7.2.2) made use of some of its special properties; in particular, we used the fact that the germ conjugacy preserves the post-critical orbits. However, in general, a conjugacy between two polynomial parabolic germs has no reason to preserve the post-critical orbits (germ conjugacies are defined locally, and post-critical orbits are global objects). Motivated by this, we asked some general local-global questions about polynomial parabolic germs in Question 7.1.6.

In this appendix, we will address the rigidity of unicritical parabolic polynomials. More precisely, we will show that a unicritical holomorphic polynomial having a parabolic cycle with a single petal at each parabolic point is completely determined by the conformal conjugacy class of its parabolic germ or equivalently, by its Ecalle-Voronin invariants. We will also prove a similar rigidity result for unicritical antiholomorphic polynomials.

We will need the concept of extended horn maps, which are the natural maximal extensions of the horn maps (see Definition 5.1.3). We will only define it in the context of parabolic points with multiplier 1 and a single petal. For a more general description of these objects, see [BE02, §2.5].

Let p be a (parabolic) holomorphic polynomial, z_0 be such that $p^{\circ k}(z_0) = z_0$, and $p^{\circ k}(z) = z + (z - z_0)^2 + \mathcal{O}(|z - z_0|^3)$ locally near z_0 . Let U_0 be the immediate basin of attraction of z_0 . Then there exists an extended attracting Fatou coordinate $\psi^{\text{att}} : U_1 \rightarrow \mathbb{C}$ (which is a ramified covering ramified only over the pre-critical points of p in U_i) satisfying $\psi^{\text{att}}(p(z)) = \psi^{\text{att}}(z) + 1$, for every $z \in U_0$ (compare Figure C.1). Similarly, the inverse of the repelling Fatou coordinate ψ^{rep} at z_0 extends to a holomorphic map $\zeta^{\text{rep}} : \mathbb{C} \rightarrow \mathbb{C}$ satisfying

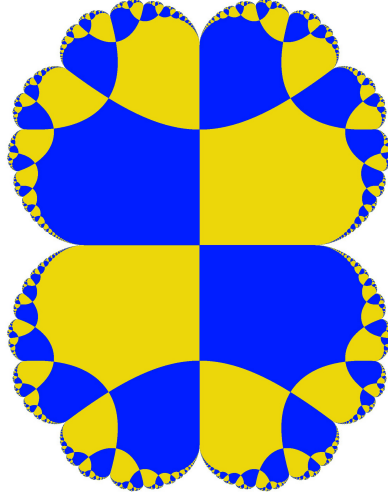


Figure C.1: The parabolic chessboard for the polynomial $z + z^2$: normalizing $\psi^{\text{att}}(-\frac{1}{2}) = 0$, each yellow tile biholomorphically maps to the upper half plane, and each blue tile biholomorphically maps to the lower half plane under ψ^{att} . The pre-critical points of $z + z^2$ or equivalently the critical points of ψ^{att} are located where four tiles meet (Figure Courtesy Arnaud Chéritat).

$p(\zeta^{\text{rep}}(\zeta)) = \zeta^{\text{rep}}(\zeta + 1)$, for every $\zeta \in \mathbb{C}$. We define D_0^+ (respectively D_0^-) to be the connected component of $(\zeta^{\text{rep}})^{-1}(U_0)$ containing an upper half plane (respectively a lower half plane). Furthermore, let $\mathcal{D}_0^\pm$ be the image of $D_0^\pm$ under the projection $\Pi : \zeta \mapsto w = \exp(2i\pi\zeta)$.

Definition C.0.9 (Extended Horn Map). *The maps $H_0^\pm := \psi^{\text{att}} \circ \zeta^{\text{rep}} : D_0^\pm \rightarrow \mathbb{C}$ are called the extended lifted horn maps for p at z_0 . They project (under Π) to the holomorphic maps $h_0^\pm : \mathcal{D}_0^\pm \rightarrow \hat{\mathbb{C}}$, which are called the extended horn maps for p at z_0 .*

We will mostly work with the horn map $h_0^+ : \mathcal{D}_0^+ \rightarrow \hat{\mathbb{C}}$. Note that $\mathcal{D}_0^+$ is the maximal domain of analyticity of the map h_0^+ . This can be seen as follows. Let $z' \in \partial U_0$, then there exists a sequence of pre-parabolic points $\{z_n\}_{n \geq 1} \subset \partial U_0$ converging to z' such that for each n , there is an arc $\gamma_n : (0, 1] \rightarrow U_0$ with $\gamma_n(0) = z_n$ satisfying the properties $\Re(\psi^{\text{att}}(\gamma_n(0, 1])) = \text{constant}$ and $\lim_{s \downarrow 0} \Im(\psi^{\text{att}}(\gamma_n(s))) = +\infty$. Therefore, for every $w' \in \partial \mathcal{D}_0^+$, there exists a sequence of points $\{w_n\}_{n \geq 1} \subset \partial \mathcal{D}_0^+$ converging to w' such that for each n , there is an arc $\Gamma_n : [0, 1] \rightarrow \mathcal{D}_0^+$ with $\Gamma_n(0) = w_n$ satisfying $\lim_{s \downarrow 0} h_0^+(\Gamma_n(s)) = 0$. It follows from the identity principle that if we could continue h_0^+ analytically in a neighborhood of w' , then h_0^+ would be identically 0, which is a contradiction

to the fact that h_0^+ is asymptotically a rotation near 0.

With these preparations, we are now ready to prove a local-global principle for parabolic germs.

Definition C.0.10. • Let $\mathcal{M}_d^{\text{cusp}}$ be the union of the set of all root points of the primitive hyperbolic components and the set of all co-root points of the multibrot set $\mathcal{M}_d$ (compare [EMS14, §3.2]). For $c_1, c_2 \in \mathcal{M}_d^{\text{cusp}}$, we write $c_1 \sim c_2$ if $z^d + c_1$ and $z^d + c_2$ are affinely conjugate; i.e. if c_2/c_1 is a $(d-1)$ -th root of unity. We denote the set of equivalence classes under this equivalence relation by $(\mathcal{M}_d^{\text{cusp}}/\sim)$.

- Let $\text{Diff}^{+1}(\mathbb{C}, 0)$ be the set of conformal conjugacy classes of holomorphic germs (at 0) fixing 0, and having multiplier $+1$ at 0.

For $c \in \mathcal{M}_d^{\text{cusp}}$, let z_c be the characteristic parabolic point and U_c be the characteristic Fatou component (of period k) of $p_c(z) = z^d + c$. Then $p_c^{\circ k}|_{N_{z_c}}$ (where N_{z_c} is a sufficiently small neighborhood of z_c) determines an element of $\text{Diff}^{+1}(\mathbb{C}, 0)$.

Theorem C.0.11 (Parabolic Germs Determine Roots of Primitive Components). *The map*

$$\begin{aligned} \bigcup_{d \geq 2} (\mathcal{M}_d^{\text{cusp}}/\sim) &\rightarrow \text{Diff}^{+1}(\mathbb{C}, 0) \\ c &\mapsto p_c^{\circ k}|_{N_{z_c}} \end{aligned}$$

is injective.

Proof. For $i = 1, 2$, let $c_i \in \mathcal{M}_{d_i}^{\text{cusp}}$, the parabolic cycle of c_i have period k_i , the characteristic parabolic points of $p_{c_i}(z) = z^{d_i} + c_i$ be z_i , and the characteristic Fatou components of p_{c_i} be U_i .

We assume that $g_1 := p_{c_1}^{\circ k_1}|_{N_{z_1}}$ and $g_2 := p_{c_2}^{\circ k_2}|_{N_{z_2}}$ are conformally conjugate by some local biholomorphism $\varphi : N_{z_1} \rightarrow N_{z_2}$. Then these two germs have the same horn map germ at 0, and hence p_{c_1} and p_{c_2} have the same extended horn map h^+ (recall that the domain of h^+ is its maximal domain of analyticity; i.e. h^+ is completely determined by the germ of the horn map at 0). If $\psi_{c_2}^{\text{att}}$ is an extended attracting Fatou coordinate for p_{c_2} at z_2 , then there exists an extended attracting Fatou coordinate $\psi_{c_1}^{\text{att}}$ for p_{c_1} at z_1 such that $\psi_{c_1}^{\text{att}} = \psi_{c_2}^{\text{att}} \circ \varphi$ in their common domain of definition. By [BE02, Proposition 4], h^+ is a ramified covering with the unique critical value $\Pi(\psi_{c_1}^{\text{att}}(c_1)) = \Pi(\psi_{c_2}^{\text{att}}(c_2))$. Note that the ramification index of h^+ over this unique critical value is $d_1 - 1 = d_2 - 1$. This shows that $d_1 = d_2$.

Furthermore, $\psi_{c_1}^{\text{att}}(c_1) - \psi_{c_2}^{\text{att}}(c_2) = n \in \mathbb{Z}$. We can normalize our attracting Fatou coordinates such that $\psi_{c_1}^{\text{att}}(c_1) = 0$ and $\psi_{c_2}^{\text{att}}(c_2) = -n$. Put $\eta := g_2^{\circ(-n)} \circ \varphi$. Then, η is a new conformal conjugacy between g_1 and g_2 . We stick to the Fatou coordinate $\psi_{c_1}^{\text{att}}$ for p_{c_1} and define a new Fatou coordinate $\tilde{\psi}_{c_2}^{\text{att}}$ for p_{c_2}

such that $\psi_{c_1}^{\text{att}} = \tilde{\psi}_{c_2}^{\text{att}} \circ \eta$ in their common domain of definition. Let N be large enough so that $p_{c_2}^{\circ k_2(N+n)}(c_2)$ is contained in the domain of definition of φ^{-1} . Now,

$$\begin{aligned} \tilde{\psi}_{c_2}^{\text{att}}(c_2) &= \tilde{\psi}_{c_2}^{\text{att}}(p_{c_2}^{\circ N k_2}(c_2)) - N \\ &= \psi_{c_1}^{\text{att}}\left(\varphi^{-1}\left(p_{c_2}^{\circ(N+n)k_2}(c_2)\right)\right) - N \\ &= \psi_{c_2}^{\text{att}}\left(p_{c_2}^{\circ(N+n)k_2}(c_2)\right) - N \\ &= \psi_{c_2}^{\text{att}}(c_2) + n + N - N \\ &= 0 \end{aligned}$$

Therefore, we have a germ conjugacy η such that the Fatou coordinates of p_{c_1} and p_{c_2} satisfy the following properties

$$\psi_{c_1}^{\text{att}} = \tilde{\psi}_{c_2}^{\text{att}} \circ \eta, \quad \psi_{c_1}^{\text{att}}(c_1) = 0, \quad \tilde{\psi}_{c_2}^{\text{att}}(c_2) = 0.$$

Since $p_{c_i}^{\circ k_i}|_{U_i}$ has a unique critical point of the same degree, one concludes (e.g. arguing as in Lemma 7.2.2) that η extends to a conformal conjugacy between $p_{c_1}^{\circ k_1}|_{U_1}$ and $p_{c_2}^{\circ k_2}|_{U_2}$. Now arguing as in Lemma 7.2.3, one sees that η extends to a conformal conjugacy between some neighborhoods of $\overline{U_i}$. The condition that c_i is a root point of a primitive hyperbolic component or a co-root point implies that z_i has exactly one attracting petal, and hence η induces a conformal conjugacy between the polynomial-like restrictions of $p_{c_1}^{\circ k_1}$ and $p_{c_2}^{\circ k_2}$ in some neighborhoods of $\overline{U_1}$ and $\overline{U_2}$ respectively.

We can now invoke [Ino11, Theorem 1] to deduce the existence of polynomials h, h_1 and h_2 such that $p_{c_1}^{\circ k_1} \circ h_1 = h_1 \circ h$, $p_{c_2}^{\circ k_2} \circ h_2 = h_2 \circ h$. In particular, we have that $\deg(p_{c_1}^{\circ k_1}) = d_1^{k_1} = d_1^{k_2} = \deg(p_{c_2}^{\circ k_2})$. Hence, $k_1 = k_2$.

The rest of the proof is similar to the reduction step of Ritt and Engstrom as in the proof of Theorem 1.2.9, so we only give a sketch. If both h_1 and h_2 are of degree 1, then we are done. If $\deg(h_i) > 1$ for some i , then arguing as in the proof of Theorem 1.2.9 (using Theorem 7.2.6 and the fact that $p_{c_i}^{\circ k_i}$ has no finite critical orbit), we conclude that $\gcd(\deg(p_{c_i}^{\circ k_i}), \deg(h_i)) > 1$. Now applying Engstrom's theorem [Eng41] (compare [Ino11, Theorem 11, Corollary 12, Lemma 13]), we obtain the existence of polynomials (of degree at least two) α_i, β_i such that, up to affine conjugacy,

$$p_{c_i}^{\circ k_i} = \alpha_i \circ \beta_i \text{ and } h = \beta_i \circ \alpha_i \tag{C.1}$$

Either $\deg(h_i) > 1$ for $i = 1$ and 2, or one of the h_i is an affine conjugacy. In either case, by looking at the multiplicities of the critical points of p and relation (C.1), we can deduce that $c_1 = c_2$ (up to affine conjugacy). $\square$

Using essentially the same ideas, one can prove a variant of the above result for polynomials of arbitrary degree, provided that the parabolic point

has exactly one petal and its immediate basin of attraction contains exactly one critical point (of possibly higher multiplicity).

Proposition C.0.12 (Unicritical Basins). *Let p_1 and p_2 be two polynomials (of any degree) satisfying $p_i(0) = 0$, and $p_i(z) = z + z^2 + \mathcal{O}(|z|^3)$ locally near 0. Let U_i be the immediate basin of attraction of p_i at 0, and p_i has exactly one critical point of multiplicity k_i in U_i . If p_1 and p_2 are (locally) conformally conjugate in some neighborhoods of 0, then $k_1 = k_2$, and there exist polynomials h , h_1 and h_2 such that $p_1 \circ h_1 = h_1 \circ h$, $p_2 \circ h_2 = h_2 \circ h$. In particular, $\deg(p_1) = \deg(p_2)$.*

The fundamental factor that makes the above proofs work is unicriticality since one can read off the conformal position of the unique critical value from the extended horn map. The next best family of polynomials, where there is some hope of applying this philosophy, is $\{f_c^{\circ 2}\}_{c \in \mathbb{C}}$.

We define $\Omega_d := \{c \in \mathbb{C} : f_c(z) = \bar{z}^d + c \text{ has a parabolic cycle of odd period with a single petal}\}$. In other words, Ω_d consists of degree d unicritical anti-polynomials having an odd periodic non-cusp parabolic cycle. As before, for $c \in \Omega_d$, the holomorphic first return map of f_c at the characteristic parabolic point determines a unique element of $\text{Diff}^{+1}(\mathbb{C}, 0)$. The first obstruction to recovering f_c from its parabolic germ comes from the following observation: if $c \in \Omega_d$ has a parabolic cycle of odd period k , then the characteristic parabolic germs of $f_c^{\circ 2k}$ and $f_{c^*}^{\circ 2k}$ are conformally conjugate by the map $\iota \circ f_c^{\circ k}$. The next theorem will show that this is, in fact, the only obstruction.

Theorem C.0.13 (Recovering Anti-polynomials from Parabolic Germs). *Let $c_1 \in \Omega_{d_1}$, $c_2 \in \Omega_{d_2}$. If the characteristic parabolic germs of $f_{c_1}(z) = \bar{z}^{d_1} + c_1$ and $f_{c_2}(z) = \bar{z}^{d_2} + c_2$ are conformally conjugate, then $d_1 = d_2$, and*

$$c_2 \in \{c_1, \omega c_1, \dots, \omega^d c_1, c_1^*, \omega c_1^*, \dots, \omega^d c_1^*\},$$

where $\omega = \exp(\frac{2\pi i}{d_1+1})$.

Proof. For $i = 1, 2$, let the parabolic cycle of $f_{c_i}(z) = \bar{z}^{d_i} + c_i$ have period k_i , the characteristic parabolic points of f_{c_i} be z_i , and the characteristic Fatou components of f_{c_i} be U_i .

We assume that $g_1 := f_{c_1}^{\circ 2k_1}|_{N_{z_1}}$ and $g_2 := f_{c_2}^{\circ 2k_2}|_{N_{z_2}}$ are conformally conjugate by some local biholomorphism $\varphi : N_{z_1} \rightarrow N_{z_2}$ (where N_{z_i} is a sufficiently small neighborhood of z_i). Then these two germs have the same horn map germ at 0, and hence $f_{c_1}^{\circ 2}$ and $f_{c_2}^{\circ 2}$ have the same extended horn map h^+ at 0 (recall that the domain of h^+ is its maximal domain of analyticity; i.e. h^+ is completely determined by the germ of the horn map at 0). If $\psi_{c_2}^{\text{att}}$ is an extended attracting Fatou coordinate for f_{c_2} at z_2 (normalized so that the attracting equator maps to the real line), then there exists an extended attracting Fatou coordinate $\psi_{c_1}^{\text{att}}$ for f_{c_1} at z_1 such that $\psi_{c_1}^{\text{att}} = \psi_{c_2}^{\text{att}} \circ \varphi$ in their

common domain of definition. Since any local conformal conjugacy preserves equators, $\psi_{c_1}^{\text{att}}$ maps the attracting equator of f_{c_1} at z_1 to the real line. By [BE02, Proposition 4], h^+ is a ramified covering with exactly two critical values. This implies that

$$\{\Pi(\psi_{c_1}^{\text{att}}(c_1)), \Pi(\psi_{c_1}^{\text{att}}(f_{c_1}^{\circ k}(c_1)))\} = \{\Pi(\psi_{c_2}^{\text{att}}(c_2)), \Pi(\psi_{c_2}^{\text{att}}(f_{c_2}^{\circ k}(c_2)))\}.$$

The ramification indices of both these critical values is $d_1 - 1 = d_2 - 1$. Hence, $d_1 = d_2$.

Case 1 ($\Pi(\psi_{c_1}^{\text{att}}(c_1)) = \Pi(\psi_{c_2}^{\text{att}}(c_2))$). We can assume, possibly after adjusting the horizontal degree of freedom of the Fatou coordinates and modifying the conformal conjugacy φ (as in the proof of Theorem C.0.11) that

$$\psi_{c_1}^{\text{att}} = \psi_{c_2}^{\text{att}} \circ \varphi, \quad \psi_{c_1}^{\text{att}}(c_1) = it = \psi_{c_2}^{\text{att}}(c_2).$$

In particular, f_{c_1} and f_{c_2} have equal critical Ecalle height and hence $f_{c_1}^{\circ 2k_1}|_{U_1}$ and $f_{c_2}^{\circ 2k_2}|_{U_2}$ are conformally conjugate. Arguing as in Lemma 7.2.2 and Lemma 7.2.3, we see that φ extends to a conformal conjugacy between $f_{c_1}^{\circ 2k_1}$ and $f_{c_2}^{\circ 2k_2}$ restricted to some neighborhoods of $\overline{U_i}$. Since c_i is a non-cusp parameter, z_i has exactly one attracting petal, and hence φ induces a conformal conjugacy between the polynomial-like restrictions of $f_{c_1}^{\circ 2k_1}$ and $f_{c_2}^{\circ 2k_2}$ in some neighborhoods of $\overline{U_1}$ and $\overline{U_2}$ respectively.

We can now invoke [Ino11, Theorem 1] to deduce the existence of polynomials h , h_1 and h_2 such that $f_{c_1}^{\circ 2k_1} \circ h_1 = h_1 \circ h$, $f_{c_2}^{\circ 2k_2} \circ h_2 = h_2 \circ h$. In particular, we have that $\deg(f_{c_1}^{\circ 2k_1}) = d_1^{2k_1} = d_1^{2k_2} = \deg(f_{c_2}^{\circ 2k_2})$. Hence, $k_1 = k_2$. Finally, applying Ritt and Engstrom's reduction steps (similar to the proof of Theorem 1.2.9), we can conclude from the semi-conjugacy relations that $c_2 \in \{c_1, \omega c_1, \dots, \omega^d c_1\}$, where $\omega = \exp(\frac{2\pi i}{d_1+1})$.

Case 2 ($\Pi(\psi_{c_1}^{\text{att}}(c_1)) = \Pi(\psi_{c_2}^{\text{att}}(f_{c_2}^{\circ k}(c_2)))$). Since φ is a conformal conjugacy between $f_{c_1}^{\circ 2k_1}|_{N_{z_1}}$ and $f_{c_2}^{\circ 2k_2}|_{N_{z_2}}$, $\tilde{\varphi} := \iota \circ f_{c_2}^{\circ k_2} \circ \varphi$ is a conformal conjugacy between the characteristic parabolic germs of $f_{c_1}^{\circ 2k_1}$ and $f_{c_2}^{\circ 2k_2}$. Let $\psi_{c_2}^{\text{att}}$ be an extended attracting Fatou coordinate for $f_{c_2}^*$ at z_2^* such that $\psi_{c_2}^{\text{att}} \circ \iota \circ f_{c_2}^{\circ k_2} = \psi_{c_2}^{\text{att}}$ in their common domain of definition. Therefore,

$$\begin{aligned} \psi_{c_1}^{\text{att}} &= \psi_{c_2}^{\text{att}} \circ \varphi \\ &= \psi_{c_2}^{\text{att}} \circ \iota \circ f_{c_2}^{\circ k_2} \circ \varphi \\ &= \psi_{c_2}^{\text{att}} \circ \tilde{\varphi} \end{aligned}$$

in their common domain of definition.

Moreover, a simple computation shows that

$$\Pi(\psi_{c_1}^{\text{att}}(c_1)) = \Pi(\psi_{c_2}^{\text{att}}(c_2^*))$$

The situation now reduces to that of Case (1), and a similar argumentation shows that $c_2 \in \{c_1^*, \omega c_1^*, \dots, \omega^d c_1^*\}$, where $\omega = \exp(\frac{2\pi i}{d_1+1})$. $\square$

Appendix D

Singularities of $\text{Per}_n(1)$: Some Examples

The parabolic arcs of the multicorns have played a pivotal role in most of this thesis. In Lemma 4.1.3, we embedded the parabolic arcs in certain dynamically defined algebraic curves, known as $\text{Per}_n(1)$. They were first considered by Milnor [Mil92, Mil00a].

Definition D.0.14. *Let $\mathcal{F}_\Lambda = \{f_\lambda\}_{\lambda \in \Lambda}$ be a holomorphic family of holomorphic polynomials of degree $d \geq 2$, parametrized by some complex manifold Λ . The algebraic set $\text{Per}_n(1)$ is defined as the set of parameters in Λ such that f_λ has a parabolic cycle of period n and multiplier 1. In algebraic terms,*

$$\text{Per}_n(1) := \{\lambda \in \Lambda : \text{disc}_z(f_\lambda^{\circ n}(z) - z) = 0\},$$

Observe that our definition does not ensure that for every $\lambda \in \text{Per}_n(1)$, the corresponding polynomial f_λ has a periodic orbit of *exact* period n with multiplier 1. Indeed, our definition allows $\text{Per}_n(1)$ to contain all parameters such that the corresponding polynomials possess a n' -periodic orbit (where $n'|n$) with multiplier a n/n' -th root of unity. However, parameters having a n' -periodic orbit (where $n'|n$) with multiplier a n/n' -th root of unity determine a polynomial that divides $\text{disc}_z(f_\lambda^{\circ n}(z) - z)$, and one can factor them out from $\text{disc}_z(f_\lambda^{\circ n}(z) - z)$ to obtain an algebraic curve consisting precisely of those maps with an n -cycle with multiplier 1. However, we assure the readers that this ambiguity in defining $\text{Per}_n(1)$ will not be of importance to us since we will mostly be interested in local properties (e.g. singularity) that are not affected by the additional components of $\text{Per}_n(1)$.

In what follows, we will look at some more examples of $\text{Per}_n(1)$ (in various families of polynomials), and will try to understand the nature of their singularities as well as the ‘dynamical’ behavior of these singular parameters. It is not our aim to prove any precise theorem here, we only intend to investigate some natural examples and to give heuristic explanations of the phenomena,

which should pave the way for a more general understanding of the topology of these algebraic sets.

1. We first consider families $\mathcal{F}$ such that Λ is a 2 complex-dimensional manifold, a generic f_λ has two infinite critical orbits, and $\text{Per}_n(1)$ is a complex curve. For any $\lambda_0 \in \text{Per}_n(1)$, we can write $f_{\lambda_0}^{\circ n}$ in local coordinates in a neighborhood of a parabolic periodic point as $f_{\lambda_0}^{\circ n}(z) = z + z^{k+1} + O(|z|^{k+2})$. Such a parabolic point has k petals and each petal attracts at least one infinite critical orbit. Since every map in our family has at most 2 infinite critical orbits, it follows that $k \in \{1, 2\}$. Furthermore, any f_λ has at most two disjoint parabolic cycles (since there are at most two infinite critical orbits). We, therefore, have to consider the following three cases.

(a) (Unique parabolic cycle with $k = 1$: non-singular point). Let U be a sufficiently small neighborhood of λ_0 in Λ . The double root of $f_{\lambda_0}^{\circ n} - \text{id}$ splits into two simple roots in $U \setminus \text{Per}_n(1)$. On a double cover B over U , ramified only over $U \cap \text{Per}_n(1)$, we can follow these two simple periodic points holomorphically as functions z_1 and z_2 . In B , $U \cap \text{Per}_n(1)$ corresponds to the smooth (complex 1-dimensional) analytic set $\{z_1 = z_2\}$ (some care is needed here; if $\tilde{a}, \tilde{b}$ are the local coordinates on B , one needs to show that the map $(\tilde{a}, \tilde{b}) \mapsto (z_1(\tilde{a}, \tilde{b}), z_2(\tilde{a}, \tilde{b}))$ is a local biholomorphism). This suggests that λ_0 is a non-singular point of $\text{Per}_n(1)$.

(b) (Unique parabolic cycle with $k = 2$: ordinary cusp). Let U be a sufficiently small neighborhood of λ_0 in Λ . The triple root of $f_{\lambda_0}^{\circ n} - \text{id}$ splits into three simple roots in $U \setminus \text{Per}_n(1)$. On a triple cover B over U , ramified only over $U \cap \text{Per}_n(1)$, we can follow these three simple periodic points holomorphically as functions z_1, z_2 and z_3 . In B , $U \cap \text{Per}_n(1)$ corresponds to the complex 1-dimensional analytic set $\{z_1 = z_2\} \cup \{z_1 = z_3\} \cup \{z_3 = z_2\}$, and the parameter λ_0 corresponds to $\{z_1 = z_2 = z_3\}$. This analytic set is not regular at $\{z_1 = z_2 = z_3\}$: two of the three conditions $\{z_i = z_j\}$ determine the same curve, and hence, the analytic set $\{z_1 = z_2\} \cup \{z_1 = z_3\} \cup \{z_3 = z_2\}$ has two coincident tangent lines (or a tangent line of multiplicity 2) at $\{z_1 = z_2 = z_3\}$. This suggests that λ_0 is an ordinary cusp singularity of $\text{Per}_n(1)$.

(c) (Two parabolic cycles each with $k = 1$: ordinary double point). Let U be a sufficiently small neighborhood of λ_0 in Λ . Then $f_{\lambda_0}^{\circ n} - \text{id}$ has two (distinct) double roots. When λ_0 is perturbed in $U \setminus \text{Per}_n(1)$, these two double roots split into two pairs of simple roots. On a double cover B over U , ramified only over $U \cap \text{Per}_n(1)$, we can follow these two pairs of simple periodic points as two pairs of holomorphic functions $\{z_1, z_2\}$ and $\{z_3, z_4\}$. In B , $U \cap \text{Per}_n(1)$ corresponds to the complex 1-dimensional analytic set $\{z_1 = z_2\} \cup \{z_3 = z_4\}$, and the parameter λ_0 corresponds to the point of self-intersection $\{z_1 = z_2\} \cap \{z_3 = z_4\}$. In other words, $U \cap \text{Per}_n(1)$ corresponds to the union of two (different) branches given by $\{z_1 = z_2\}$ and $\{z_3 = z_4\}$, and λ_0 corresponds to the point where these two branches intersect (transversally). Therefore, λ_0 is an ordinary double point (with non-vanishing Hessian and two distinct tangent lines) of $\text{Per}_n(1)$.

Examples. i) In Subsection 4.1.1, we discussed the properties of $\text{Per}_1(1)$ of the family $\{(z^2 + a)^2 + b\}_{a,b \in \mathbb{C}}$: the only singularities of this algebraic curve correspond to the parameters with double parabolic points, and each of these singularities is of the form $x^2 - y^3$ at $(0,0)$. Every other point of this curve is non-singular.

ii) Let us look at the family of monic centered cubic polynomials $\mathcal{F} := \{f_{a,b}(z) = z^3 - 3az + b, a, b \in \mathbb{C}\}$. For this family, we have

$$\begin{aligned}\text{Per}_1(1) &= \{(a, b) \in \mathbb{C}^2 : \text{disc}_z(f_{a,b}(z) - z) = 0\} \\ &= \{(a, b) \in \mathbb{C}^2 : 4 + 36a + 108a^2 + 108a^3 - 27b^2 = 0\}\end{aligned}$$

Once again, the only singular point of this curve is $(-\frac{1}{3}, 0)$, which is the only double parabolic point of $\mathcal{F}$. In fact, $(-\frac{1}{3}, 0)$ is a double point of $\text{Per}_1(1)$ with vanishing Hessian. To better visualize this singularity, we may consider the following dynamically defined morsification. The algebraic curve consisting of all parameters (a, b) for which $f_{a,b}$ has a fixed point of multiplier r is referred to as $\text{Per}_1(r)$. More precisely, for $r \in \mathbb{C}$, we define

$$\begin{aligned}h_r(a, b) &:= \text{Res}_z(f_{a,b}(z) - z, f'_{a,b}(z) - r), \\ &= 108a^3 + 108a^2 - 27b^2 - 9a(r-3)(r+1) + r(r-3)^2, \\ \text{Per}_1(r) &:= \{(a, b) \in \mathbb{C}^2 : h_r(a, b) = 0\},\end{aligned}$$

where $\text{Res}_z(g_1, g_2)$ of two polynomial g_1 and g_2 in one complex variable denotes the resultant of g_1 and g_2 .

Then h_r is a morsification of h_1 in the sense that for $r \neq 1$, h_r has two non-degenerate (i.e. with non-vanishing Hessian) critical points at $(\frac{r-3}{6}, 0)$ and $(-\frac{r+1}{6}, 0)$. Let us try to understand the formation of the cusp-type singularity of h_1 at $(-1/3, 0)$ in the following dynamical fashion. For $(a_1, b_1) = (\frac{r-3}{6}, 0)$, f_{a_1, b_1} has two distinct fixed points of multiplier r . Hence, $(\frac{r-3}{6}, 0)$ is the point of intersection of two transversal branches of $\text{Per}_1(r)$, and is a node-type singular point of $\text{Per}_1(r)$ (i.e. a double point with two distinct tangent lines). At the other critical point $(a_2, b_2) = (-\frac{r+1}{6}, 0)$ of h_r , f_{a_2, b_2} has two distinct fixed points of multiplier $(2-r)$. As $r \rightarrow 1$, these two critical points (each having Milnor number 1) of h_r coalesce to form the critical point $(-1/3, 0)$ (with Milnor number 2) of h_1 , and all the fixed points coalesce to form a triple fixed point of $f_{-1/3, 0}$.

iii) We now consider $\text{Per}_1(1)$ of the family $\{(z^3 + a)^3 + b\}_{a,b \in \mathbb{C}}$. The parameter $(a_0, b_0) \approx (0.7698 + 0.7698i, 0.7698 - 0.7698i)$ is a co-root of a hyperbolic component of period 2 of $\mathcal{M}_3^*$. The polynomial $(z^3 + a_0)^3 + b_0$ has two simple parabolic fixed points, and (a_0, b_0) is an ordinary double point (with non-vanishing Hessian and two distinct tangent lines) of $\text{Per}_1(1)$ of this family.

The local topology of $\text{Per}_n(1)$ near these singularities can be studied via the singularity link and the Milnor fibration (compare [Mil68]). In fact, in the first two examples above, $\text{Per}_n(1)$ is locally (near the singularity) a cone over

the trefoil knot. In the third example, $\text{Per}_n(1)$ is locally a cone over the Hopf link.

2. We now look at families with $m \geq 3$ infinite critical orbits such that Λ is complex m -dimensional. For simplicity, we will restrict ourselves to the case $m = 3$ (this case is already more complicated than the case where Λ is complex 2-dimensional). As above, for any $\lambda_0 \in \text{Per}_n(1)$, we can write $f_{\lambda_0}^{\text{on}}$ in local coordinates in a neighborhood of a parabolic point as $f_{\lambda_0}^{\text{on}}(z) = z + z^{k+1} + O(|z|^{k+2})$ with $k \in \{1, 2, 3\}$. For ease of exposition, let us work with the family $\mathcal{F} := \{f_{a,b,c}(z) = z^4 + az^2 + bz + c, a, b, c \in \mathbb{C}\}$ of monic centered quartic polynomials. Then,

$$\begin{aligned} \text{Per}_1(1) &= \{(a, b, c) \in \mathbb{C}^3 : \text{disc}_z(z^4 + az^2 + bz + c - z) = 0\} \\ &= \{(a, b, c) \in \mathbb{C}^3 : -27 - 4a^3 + 108b + 8a^3b \\ &\quad - 162b^2 - 4a^3b^2 + 108b^3 - 27b^4 + 144ac + 16a^4c \\ &\quad - 288abc + 144ab^2c - 128a^2c^2 + 256c^3 = 0\} \end{aligned}$$

To take a closer look at the different types of singularities, we need to consider four sub-cases.

(a) (Unique parabolic fixed point with $k = 1$: non-singular point). The treatment of this case is similar to that of Case 1(a).

(b) (Unique parabolic fixed point with $k = 3$: triple point). The only parameter for which $f_{a,b,c}$ has a parabolic cycle with three petals is $(0, 1, 0)$. A simple yet lengthy computation shows that $(0, 1, 0)$ is a singular point of $\text{Per}_1(1)$; in particular, it is a triple point with three coincident tangents (or a tangent of multiplicity 3). A heuristic reasoning for this kind of singularity can be given along the same lines of Case 1(b).

(c) (Non-isolated singularities). Since $\text{Per}_1(1)$ is a two-dimensional algebraic set, it is perhaps not surprising that it has non-isolated singularities along a complex one-dimensional algebraic subset. These non-isolated singular parameters are given by the intersection of the algebraic sets

$$\begin{aligned} V := \{(a, b, c) \in \mathbb{C}^3 : \text{sRes}_0(f_{a,b,c}(z) - z, f'_{a,b,c}(z) - 1) = \\ \text{sRes}_1(f_{a,b,c}(z) - z, f'_{a,b,c}(z) - 1) = 0\} \end{aligned}$$

This is, in fact, the intersection of $\text{Per}_1(1)$ with $\{(a, b, c) \in \mathbb{C}^3 : 36 + 8a^3 - 72b + 36b^2 - 32ac = 0\}$. Each point of V is a singular point of $\text{Per}_1(1)$.

As is clear from the defining property, for each $(a, b, c) \in V$, the corresponding polynomials $f_{a,b,c}(z) - z$ and $f'_{a,b,c}(z) - 1$ have at least two common factors. The case where they have three common factors is already covered in the previous case, so we will now be concerned with the case $\gcd(f_{a,b,c}(z) - z, f'_{a,b,c}(z) - 1) = 2$. This can happen in two different ways.

i) (Two parabolic fixed points each with $k = 1$). Each parameter of the form $(a, 1, \frac{a^2}{4})$ ($a \in \mathbb{C}^*$) belongs to V , and the corresponding polynomial $f_{a,1,\frac{a^2}{4}}$

has two distinct simple parabolic fixed points $\pm\sqrt{-\frac{a}{2}}$. This case is similar to Case 1(c); the singularities are formed by the transversal intersection of two branches of the algebraic set. Hence, each $(a, 1, \frac{a^2}{4})$ ($a \in \mathbb{C}^*$) is a double point, and there are two tangents at each such singularity.

ii) (Unique parabolic fixed point with $k = 2$). For each $(a, b, c) \in V \setminus (\{0, 1, 0\} \cup \{(a, 1, \frac{a^2}{4}) : a \in \mathbb{C}^*\})$, $f_{a,b,c}$ has a unique parabolic fixed point with two petals. Each such parameter is a double point with a single tangent of multiplicity 2. A heuristic reasoning for this kind of singularity can be given along the same lines of Case 1(b).

Let us summarize our observations regarding the singular locus of $\text{Per}_1(1)$ of degree 4 monic centered polynomials. The singular locus V admits a natural stratification $V = V_0 \cup V_1 \cup V_2$, where

$$\begin{aligned} V_0 &= \{0, 1, 0\}, \\ V_1 &= \{(a, b, c) \in \mathbb{C}^3 : b = 1, a^2 - 4c = 0\} \setminus \{0, 1, 0\}, \\ V_2 &= \{(a, b, c) \in \mathbb{C}^3 : 8a^3 + 27(1 - b)^2 = 0, a^2 + 12c = 0\} \setminus \{0, 1, 0\}. \end{aligned}$$

Here, $\{0, 1, 0\}$ is a triple point of $\text{Per}_1(1)$ with a tangent of multiplicity 3, each point of V_1 is a double point with two distinct tangents, and each point of V_2 is a double point with a tangent of multiplicity 2. Therefore, the Milnor fibrations at any two points of V_i have the same fibration type.

In general, we should ask the following questions, which are clearly motivated by the preceding analysis.

Question D.0.15. *Let Λ be a complex m -dimensional manifold and $\mathcal{F}_\Lambda = \{f_\lambda\}_{\lambda \in \Lambda}$ be a holomorphic family of holomorphic polynomials of degree $d \geq 2$, parametrized by Λ .*

1. *Does every singularity of $\text{Per}_n(1)$ occur either due to orbit mergers (giving rise to cusp-type singularities) or due to transverse intersection of more than one branches, each defining a periodic orbit of multiplier 1 (giving rise to node-type singularities)?*
2. *Can we classify the types of singularities of $\text{Per}_n(1)$ in terms of dynamical properties of the singular parameters?*
3. *Study the topology of $\text{Per}_n(1)$ near its singularities, especially near the non-isolated ones.*

Bibliography

- [BBM15] Araceli Bonifant, Xavier Buff, and John Milnor. On antipode preserving cubic maps. URL: <http://www.math.sunysb.edu/~jack/bbm.pdf>, 2015.
- [BDH04] Gamaliel Ble, Adrien Douady, and Christian Henriksen. Round annuli. *Contemporary Mathematics: In the Tradition of Ahlfors and Bers, III*, 355:71–76, 2004.
- [BE02] Xavier Buff and Adam L. Epstein. A parabolic Pommerenke-Levin-Yoccoz inequality. *Fund. Math.*, 172:249–289, 2002.
- [BE07] Xavier Buff and Adam L. Epstein. From local to global analytic conjugacies. *Ergodic Theory and Dynamical Systems*, 27:1073–1094, 2007.
- [Bea91] Alan F. Beardon. *Iteration of Rational Functions*, volume 132 of *Graduate Texts in Mathematics*. Springer, 1991.
- [BH88] Bodil Branner and John H. Hubbard. The iteration of cubic polynomials, part I: The global topology of parameter space. *Acta Math.*, 160:143–206, 1988.
- [BPR06] Saugata Basu, Richard Pollack, and Marie-Francoise Coste Roy. *Algorithms in Real Algebraic Geometry*. Springer, Berlin, 2nd edition, 2006.
- [BS02] Henk Bruin and Dierk Schleicher. Symbolic dynamics of quadratic polynomials. Technical Report 7, Institut Mittag-Leffler, 2001–2002.
- [CFG15] Jordi Canela, Núria Fagella, and Antonio Garijo. On a family of rational perturbations of the doubling map. URL: <http://arxiv.org/abs/1501.03647>, 2015.
- [CG93] L. Carleson and T. W. Gamelin. *Complex Dynamics*. Springer, Berlin, 1993.

-
- [CHRC89] W. D. Crowe, R. Hasson, P. J. Rippon, and P. E. D. Strain Clark. On the structure of the Mandelbar set. *Nonlinearity*, 2, 1989.
- [DF03] David S. Dummit and Richard M. Foote. *Abstract Algebra*. John Wiley and Sons, Inc., 3rd edition, 2003.
- [DF07] Romain Dujardin and Charles Favre. Distribution of rational maps with a preperiodic critical point. URL: <http://arxiv.org/pdf/math/0601612v3.pdf>, 2007.
- [DH85a] Adrien Douady and John H. Hubbard. *Étude dynamique des polynômes complexes I*. Publications Mathématiques d’Orsay. Université de Paris-Sud, Département de Mathématiques, Orsay, 1984 - 1985.
- [DH85b] Adrien Douady and John H. Hubbard. On the dynamics of polynomial-like mappings. *Ann. Sci. Ec. Norm. Sup.*, 18:287–343, 1985.
- [Dob11] Neil Dobbs. Nice sets and invariant densities in complex dynamics. *Math. Proc. Cambridge Philos. Soc.*, 150:157–165, 2011.
- [Dou83] Adrien Douady. Systèmes dynamiques holomorphes. *Seminaire Bourbaki, Astérisque*, 599:39–63, 1983.
- [Dou94] Adrien Douady. Does a Julia set depend continuously on the polynomial? *Complex dynamical systems, Proc. Sympos. Appl. Math.*, 49:91–138, 1994.
- [DU91] Manfred Denker and Mariusz Urbański. Hausdorff and conformal measures on Julia sets with a rationally indifferent periodic point. *J. London Math. Soc.*, 43:107–118, 1991.
- [Duj09] Romain Dujardin. Cubic polynomials: A measurable view on parameter space. In Dierk Schleicher, editor, *Complex Dynamics: Families and Friends*, pages 451–490. A K Peters, Ltd., 2009.
- [Duj14] Romain Dujardin. Bifurcation currents and equidistribution in parameter space. In Araceli Bonifant, Mikhail Lyubich, and Scott Sutherland, editors, *Frontiers in Complex Dynamics: In Celebration of John Milnor’s 80th Birthday*, pages 515–566. Princeton University Press, 2014.
- [Eca75] Jean Ecale. Théorie itérative : introduction à la théorie des invariants holomorphes. *J. Math. Pures Appl. (9)*, 54:183–25, 1975.

- [EMS14] Dominik Eberlein, Sabyasachi Mukherjee, and Dierk Schleicher. Rational parameter rays of the multibrot sets. URL: <http://arxiv.org/pdf/1410.6729v1.pdf>, submitted, 2014.
- [Eng41] H. T. Engstrom. Polynomial substitutions. *Amer. J. Math.*, 63:249–255, 1941.
- [FG13] Charles Favre and Thomas Gauthier. Distribution of postcritically finite polynomials. URL: <http://arxiv.org/pdf/1302.0810.pdf>, 2013.
- [GM93] Lisa R. Goldberg and John Milnor. Fixed points of polynomial maps II: Fixed point portraits. *Ann. Scient. École Norm. Sup.*, 4^e série, 26:51–98, 1993.
- [GM05] John B. Garnett and Donald D. Marshall. *Harmonic Measure*. Cambridge University Press, New York, 2005.
- [Gol92] Lisa R. Goldberg. Fixed points of polynomial maps I: Rotation subsets of the circles. *Ann. Scient. École Norm. Sup.*, 4^e série, 25:679–685, 1992.
- [GŚ97] Jacek Graczyk and Grzegorz Świątek. Generic hyperbolicity in the logistic family. *Annals of Mathematics*, 146:1–52, 1997.
- [HS14] John Hubbard and Dierk Schleicher. Multicorns are not path connected. In *Frontiers in Complex Dynamics: In Celebration of John Milnor's 80th Birthday*, pages 73–102. Princeton University Press, 2014.
- [IK12] Hiroyuki Inou and Jan Kiwi. Combinatorics and topology of straightening maps, I: Compactness and bijectivity. *Advances in Mathematics*, 231:2666–2733, 2012.
- [IM14] Hiroyuki Inou and Sabyasachi Mukherjee. Non-landing parameter rays of the multicorns. URL: <http://arxiv.org/pdf/1406.3428v2.pdf>, submitted, 2014.
- [Ino09] Hiroyuki Inou. Combinatorics and topology of straightening maps II: Discontinuity. URL: <http://arxiv.org/abs/0903.4289>, 2009.
- [Ino11] Hiroyuki Inou. Extending local analytic conjugacies. *Trans. Amer. Math. Soc.*, 363:331–343, 2011.
- [Ino14] Hiroyuki Inou. Self-similarity for the tricorn. URL: <http://arxiv.org/pdf/1411.3081.pdf>, 2014.

-
- [Kiw97] Jan Kiwi. Rational rays and critical portraits of complex polynomials. URL: <http://arxiv.org/abs/math/9710212>, 1997.
- [KN04] Yohei Komori and Shizuo Nakane. Landing property of stretching rays for real cubic polynomials. *Conformal Geometry and Dynamics*, 8:87–114, 2004.
- [KSvS07] O. Kozlovski, W. Shen, and S. van Strien. Density of hyperbolicity in dimension one. *Annals of Mathematics*, 166:145–182, 2007.
- [Lav89] Pierre Lavaurs. *Systèmes dynamiques holomorphes: explosion de points périodiques paraboliques*. PhD thesis, Université de Paris-Sud Centre d’Orsay, 1989.
- [Lev90] Genadi M. Levin. Theory of iterations of polynomial families in the complex plane. *Journal of Soviet Mathematics*, 52:3512–3522, 1990.
- [LS96] Eike Lau and Dierk Schleicher. Symmetries of fractals revisited. *The Mathematical Intelligencer*, 18:45–51, 1996.
- [Lyu97] Mikhail Lyubich. Dynamics of quadratic polynomials. I, II. *Acta Math.*, 178:185–247, 247–297, 1997.
- [McM00] Curtis T. McMullen. Hausdorff dimension and conformal dynamics II: Geometrically finite rational maps. *Commentarii Mathematici Helvetici*, 75:535–593, 2000.
- [Mil68] John Milnor. *Singular Points of Complex Hypersurfaces*. Annals of Mathematics Studies. Princeton University Press, New Jersey, 1968.
- [Mil92] John Milnor. Remarks on iterated cubic maps. *Experimental Mathematics*, 1:5–24, 1992.
- [Mil00a] John Milnor. On rational maps with two critical points. *Experiment. Math.*, 9:333–411, 2000.
- [Mil00b] John Milnor. Periodic orbits, external rays and the Mandelbrot set. *Astérisque*, 261:277–333, 2000.
- [Mil06] John Milnor. *Dynamics in one complex variable*. Princeton University Press, New Jersey, 3rd edition, 2006.
- [MNS14] Sabyasachi Mukherjee, Shizuo Nakane, and Dierk Schleicher. On Multicorns and Unicorns II: bifurcations in spaces of antiholomorphic polynomials. URL: <http://arxiv.org/abs/1404.5031>, to appear in ‘Ergodic Theory and Dynamical systems’, 2014.

- [MP12] John Milnor and Alfredo Poirier. Hyperbolic components. URL: <http://arxiv.org/abs/1205.2668>, 2012.
- [MSS83] R. Mané, P. Sad, and D. Sullivan. On the dynamics of rational maps. *Ann. Sci. École Norm. Sup.*, 16(2):193–217, 1983.
- [Muk14] Sabyasachi Mukherjee. Real-analyticity of Hausdorff dimension of Julia sets along the parabolic arcs of the multicorns. URL: <http://arxiv.org/abs/1410.1180v1>, 2014.
- [Muk15] Sabyasachi Mukherjee. Orbit portraits of unicritical antiholomorphic polynomials. *Conformal Geometry and Dynamics of the AMS*, 19:35–50, 2015.
- [Nak93a] Shizuo Nakane. Connectedness of the Tricorn. *Ergodic Theory and Dynamical Systems*, 13:349–356, 1993.
- [Nak93b] Shizuo Nakane. On quasiconformal equivalence on the boundary of the Tricorn. *Structure and bifurcation of dynamical systems, World Scientific.*, pages 154–167, 1993.
- [NS96] Shizuo Nakane and Dierk Schleicher. The nonarcwise-connectedness of the tricorn. In *Problems concerning complex dynamical systems (Japanese)*, number 959 in *Sūrikaiseikikenkyūsho Kōkyūroku*, pages 73–83. Kyoto University, Research Institute for Mathematical Sciences, Kyoto, 1996.
- [NS03] Shizuo Nakane and Dierk Schleicher. On Multicorns and Unicorns I: Antiholomorphic dynamics, hyperbolic components and real cubic polynomials. *International Journal of Bifurcation and Chaos*, 13:2825–2844, 2003.
- [Poi93] Alfredo Poirier. On postcritically finite polynomials, part 1: Critical portraits. URL: <http://arxiv.org/abs/math/9305207>, 1993.
- [PR08] Carsten L. Petersen and Pascale Roesch. The Yoccoz combinatorial analytic invariant. In *Holomorphic dynamics and renormalization: A Volume in Honour of John Milnor's 75th Birthday*, pages 145–176. AMS and The Fields Institute, 2008.
- [Rit22] J. F. Ritt. Prime and composite polynomials. *Trans. Amer. Math. Soc.*, 23:5166, 1922.
- [RL07] Juan Rivera-Letelier. A connecting lemma for rational maps satisfying a no-growth condition. *Ergodic Theory Dynam. Systems*, 27:595–636, 2007.

-
- [Rud87] Walter Rudin. *Real and Complex Analysis*. McGraw-Hill Book Company, Inc., New York, 3 edition, 1987.
- [Sch98] Dierk Schleicher. On fibers and renormalization of Julia sets and Multibrot sets. URL: <http://www.math.sunysb.edu/preprints/>, 1998.
- [Sch00] Dierk Schleicher. Rational parameter rays of the Mandelbrot set. *Astérisque*, 261:405–443, 2000.
- [Shi00] Mitsuhiro Shishikura. Bifurcation of parabolic fixed points. In *The Mandelbrot Set, Theme and Variations*, London Mathematical Society Lecture Note Series (No. 274), pages 325–364. Cambridge University Press, 2000.
- [SU14] Bartłomiej Skorulski and Mariusz Urbański. Finer fractal geometry for analytic families of conformal dynamical systems. *Dynamical Systems*, 29:369–398, 2014.
- [Urb03] Mariusz Urbański. Measures and dimensions in conformal dynamics. *Bull. Amer. Math. Soc.*, 40:281–321, 2003.
- [Vor81] S. M. Voronin. Analytic classification of germs of conformal mappings $(\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$. *Funktsional. Anal. i Prilozhen*, 15:1–17, 1981.
- [Wal79] Peter Walters. A variational principle for the pressure of continuous transformations. *Amer. J. Math.*, 97:937–971, 1979.
- [Wal82] Peter Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer, 1982.
- [Wal04] C. T. C. Wall. *Singular Points of Plane Curves*, volume 63 of *London Mathematical Society Student Texts*. Cambridge University Press, 2004.
- [Win90] R. Winters. *Bifurcations in families of antiholomorphic and bi-quadratic maps*. PhD thesis, Boston University, 1990.
- [Zin00] Michel Zinsmeister. *Thermodynamic formalism and holomorphic dynamical systems*, volume 2 of *SMF/AMS Texts and Monographs*. AMS and Société Mathématique de France, 2000.